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Reduced order H_∞ controller design

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REDUCED ORDER H_∞ CONTROLLER DESIGN

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Dedicated to my father

ABSTRACT

The problem formulation and the development of the H_∞ optimal control theory are reviewed. Various H_∞ robust controller design techniques, including both state and output feedback cases, together with a brief comparison and some remarks on these methodologies are presented. Also included are a review of the controller order reduction alternatives and various balanced truncation model reduction methods.

A new model reduction technique with reduced error bound is proposed. This proposed technique introduces a new parameter which can be used to select the spectral norm of the model reduction error at low and high frequency ranges according to the designer's needs. Moreover, the upper bound of the model reduction error resulted from this proposed technique is significantly lower than those attainable by any of the existing balanced truncation techniques.

The concept of combined state and output feedback H_∞ controller is also introduced. This proposed method yields an H_∞ controller with order less than the order of the generalized plant. Corresponding to the modified closed-loop structure resulted from this method, the criteria for performance evaluation, namely the formulae for the sensitivity and the closed-loop transfer function are derived. Also, the definition of sensitivity of a multi-variable closed-loop system is generalized to the state-feedback case.

To demonstrate these proposed techniques, two practical examples are presented. The formulae derived together with the new theorems and lemmas are all verified.

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December, 1990 Seng Chu Chow

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I. INTRODUCTION

The H_∞ optimal control theory which was first introduced to the control literature by Zames (1981) offers a new design philosophy for multivariable control systems by combining the classical frequency domain idea with the modern time domain optimization approach. Unlike the Linear Quadratic Gaussian (LQG) or the Kalman filter problems where the power spectrum of the noise input is assumed known, the H_∞ control problem is to design a controller to optimize the closed-loop performance for the worst exogeneous signals. In fact, the LQG problem, as proven by Dailey (1990), can be seen as a special case of the H_∞ problem in which the norm-bounding scalar γ tends to infinity. Since γ bounds the L_∞-norm of the error transfer function, by keeping it to its minimum as is done in the H_∞ control problem, one can expect the H_∞ controller to perform better than the LQG controller. This conclusion is proven by numerical examples, for instance, see Maciejowski (1989).

The original formulation of the H_∞ optimal control problem was developed in an input-output setting and the available solution techniques at that time involved complex mathematical concepts such as operator-theoretic mathods. The first state-space solution to the H_∞ control problem was presented by Doyle (1984). In that method, the H_∞ problem is first transformed to an equivalent model-matching problem by utilizing the Q-parameterization technique, the general distance problem which involves mixed Hankel-Toeplitz operators, and the Nehari problem for direct solution. Unfortunately, the computational burden associated with this technique together with the dimension and complexity of the resulting controller have greatly reduced the usefulness of this method in the practical sense.

However, substantial progress in this field has been made in recent years and important contributions from researchers around the world have virtually eliminated the above-mentioned drawback of H_∞ control. Glover (1984) with his Hankel Approximation approach, Hung (1989a,b) with his H_∞ interpolation theory, Kimura (1989) with his J-lossless conjugation technique and Glover and Doyle (1988) with their modified and simplified '1984 method' have all solved the H_∞ control problem in input-output setting. Each of these solutions requires a relatively simple computation procedure and the order of the resulting controller is identical to that of the generalized plant.

The pioneering work in H_∞ control for the full-state feedback case is probably by Petersen (1987). Since then, there has been considerable interest in this topic and several researchers worldwide have used different techniques to solve this problem. Petersen (1987) first solved the problem but his method requires the repeated solution of an algebraic Riccati equation for each value of γ. This method was modified and slightly generalized also by Petersen (1989). A new approach utilizing the Riccati inequality was proposed by Scherer (1990) while a new problem of simultaneous H_2/H_∞ optimal control was introduced and solved by Rotea and Khargonekar (1990) for the state-feedback case.

Each of the solutions to the H_∞ control problem mentioned above is unique in the sense that they make different assumptions and utilize different mathematical tools. The various distinctive features of each method are discussed in Section C of Chapter IV.

The balanced truncation model reduction technique has gained significant popularity since its introduction by Moore in 1981. Recently, Prakash and Rao (1989) have shown that the model reduction error associated with Moore's method is large at low frequency while it reduces to zero at high frequency. This is not a desirable feature since most practical systems do operate in low frequency range with finite bandwidth. In turn, Prakash and Rao (1989) proposed a new model reduction technique that introduces a low model reduction error at low frequency while it asymptotically approaches a finite value as the radiant frequency ω tends to infinity. This thesis proposes a new balanced truncation model reduction technique which embraces both Moore's and Prakash and Rao's methods as two special cases. The design parameter introduced by this proposed technique offers an extra degree of freedom in which the relative magnitude of the model reduction error at low and high frequency ranges can be easily altered. By properly selecting a value for this parameter, the upper bound of the model reduction error can be made as low as one half of those achieved by either Moore's or Prakash and Rao's methods. Also, this new method is proved to be no worse than the two methods mentioned above.

A lower order controller is usually desirable for reason of reduced computational load and hence, significant research effort has been devoted to various model reduction techniques in both frequency and time domains. There are generally two ways to acheive a lower order controller, one is to reduce the order of the plant before designing a controller for it while the other is to design a full order controller and then reduce its dimension. Both of these ways are indirect. This thesis proposes a direct way to design a reduced order H_∞ controller by decomposing the plant into two portions, one with full-state feedback while the other with output feedback, and design a H_∞ controller for each sub-system before combining the controllers to obtain a single controller with order n_1 less than the order of the plant, where n_1 is the number of measurable states. This approach incurs a different closed-loop structure which was fully analyzed. Also, the closed-loop performance criteria, namely the sensitivity function and the closed-loop transfer function are explicitly derived.

In this high technology era and computer age, a control engineer is equipped with various well-developed and powerful computer softwares suitable for control designs such as Matrix_X, Ctrl-C and PC-Matlab. The two examples presented in Chapter VI are implemented, designed and simulated using the main software of Matrix_X together with its extended portion known as Robust Control Module.

The notations and nomenclatures used throughout this thesis are established in Chapter II where some background materials are also given. These materials are fundamental to the development of H_∞ control theory and facilitate the discussions in the subsequent chapters. Chapter III provides the problem statement as well as some formulation examples of the H_∞ optimal control problem. Some design methodologies of H_∞ controller for both output feedback and state feedback cases are presented in Sections A and B of Chapter IV, respectively. We show in Section B.4 that the definition of sensitivity for output feedback is naturally and directly extended to the state feedback case. Some remarks on static and dynamic state-feedback H_∞ controllers together with a comparison of various designing techniques are provided in Section C. Section A of Chapter V reviews several balanced truncation model reduction methods. The new model reduction technique proposed in Section B offers the designer the luxury of compromizing the spectral norm of the model reduction error between low and high frequency ranges. Section C is devoted to the discussion of the order reduction alternatives, namely, plant model reduction and controller model reduction. In section D, a new direct reduced order H_∞ controller design technique is proposed. For purposes of illustration and verification, two examples are presented in Chapter VI where the new formulae, theorems and lemmas are tested. Finally, some concluding remarks and suggestions for further research are given in Chapter VII.

II. SYSTEM NOTATIONS AND MATHEMATICAL BACKGROUND

A. NOTATIONS AND NOMENCLATURES

The state-space realization of a transfer function matrix G(s) is denoted as

$$G(s) = C(\, sI - A\,)^{-1}B + D = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = (\, A, B, C, D\,)\,. \tag{2.1}$$

In this notation, we have

$$\tilde{G}(s) = G^T(-s) = (\, -A^T, C^T, -B^T, D^T\,) \tag{2.2}$$

and

$$G^{-1}(s) = (\, A - BD^{-1}C,\ BD^{-1},\ -D^{-1}C,\ D^{-1}\,) \tag{2.3}$$

provided D^{-1} exists. A similarity transformation with transformation matrix T applied to G(s) is defined to be

$$(\, A, B, C, D\,) \rightarrow (\, TAT^{-1}, TB, CT^{-1}, D\,)\,. \tag{2.4}$$

If $G_1(s) = (\, A_1, B_1, C_1, D_1\,)$ and $G_2(s) = (\, A_2, B_2, C_2, D_2\,)$, then the cascade system of $G_1(s)\, G_2(s)$ has a combined realization given by

$$G_1(s)\, G_2(s) = \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right]. \tag{2.5}$$

For the feedback connection shown in Figure 1, if $G_1(s)$ and $G_2(s)$ have the realizations

$$G_i(s) = \left[\begin{array}{cc|c} A^i & B_1^i & B_2^i \\ \hline C_1^i & D_{11}^i & D_{12}^i \\ C_2^i & D_{21}^i & 0 \end{array}\right], \qquad i=1,2\,, \tag{2.6}$$

then the realization for the overall feedback system is given by

$$FB(\, G_1(s)\,, G_2(s)\,) = (\, A_{fb}, B_{fb}, C_{fb}, D_{fb}\,) \tag{2.7}$$

where

$$A_{fb} = \begin{bmatrix} A^1 + B_2^1 D_{11}^2 C_2^1 & B_2^1 C_1^2 \\ B_1^2 C_2^1 & A^2 \end{bmatrix},$$

$$B_{fb} = \begin{bmatrix} B_1^1 + B_2^1 D_{11}^2 D_{21}^1 & B_2^1 D_{12}^2 \\ B_1^2 D_{21}^1 & B_2^2 \end{bmatrix},$$

$$C_{fb} = \begin{bmatrix} C_1^1 + D_{12}^1 D_{11}^2 C_2^1 & D_{12}^1 C_1^2 \\ D_{21}^2 C_2^1 & C_2^2 \end{bmatrix},$$

$$D_{fb} = \begin{bmatrix} D_{11}^1 + D_{12}^1 D_{11}^2 D_{21}^1 & D_{12}^1 D_{12}^2 \\ D_{21}^2 D_{21}^1 & 0 \end{bmatrix}.$$

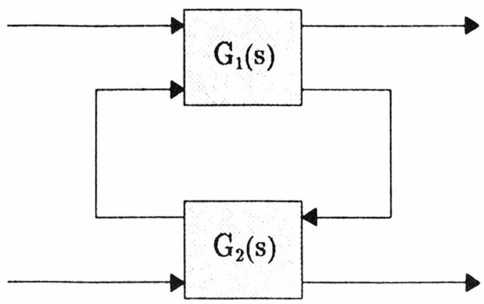

Figure 1 A feedback connection

Let H(s) be a partitioned transfer function matrix with the following realization :

$$H(s) = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \tag{2.8}$$

then the (i,j) sub-block of H(s) has a realization

$$H_{ij}(s) = C_i\,(\,sI - A\,)^{-1} B_j + D_{ij}\ . \tag{2.9}$$

We note here that the dependency on the complex frequency s will be suppressed whenever the possibility of confusion does not exist.

G(s) is said to be stable if $G(s) \in H_+$ and anti-stable if $G(s) \in H_-$. A square transfer matrix G(s) is said to be all-pass if $G(s)\cdot\tilde{G}(s) = I$. An all-pass transfer function is called inner (anti-inner) if it is stable (anti-stable). The set of all inner (anti-inner) matrices will be denoted by I_+ (I_-). A transfer matrix U(s) of size m×n satisfying $\|U(s)\|_\infty \le 1$ will be called sub-all-pass. A sub-all-pass transfer matrix which is stable is known as sub-inner. The set of all sub-inner transfer matrices is denoted by S_+. Also, we use $\mathbf{R}$ to denote the field of real numbers and $\mathbf{R}H_\infty^{m*r}$ the set of all proper, stable and rational matrices of size m×r. The set of all $\Phi \in \mathbf{R}H_\infty^{m*r}$ satisfying $\|\Phi\|_\infty < 1$ is denoted by $\mathbf{B}H_\infty^{m*r}$.

B. MATHEMATICAL PRELIMINARIES

This section serves to facilitate the subsequent discussions by presenting some fundalmental materials in H_∞ control theory. Section 1 introduces the fractional representation while Section 2 is devoted to the discussion of internal stability. The problem statement as well as the complete solution to the H_∞ interpolation problem is presented in Section 3. In Section 4, the parameterization of all stabilizing controllers is developed while the notion of model-matching is introduced in Section 5. The loop shifting procedure is expounded in Section 6 and the idea of conjugation is contained in the last section.

1. Fractional Representation : For every proper transfer function matrix G(s), there exist stable right coprime matrices U and V and left coprime matrices $\tilde{U}$ and $\tilde{V}$ such that

$$G(s) = U(s)\,V^{-1}(s) = \tilde{V}^{-1}(s)\,\tilde{U}(s)\,. \tag{2.10}$$

The transfer function matrices U and V are said to be right coprime if and only if there exist stable X and Y such that the Bezout's identity

$$XU + YV = I \tag{2.11}$$

is satisfied.

To obtain a fractional representation for a stabilizable and detectable realization $G(s) = (A, B, C, D)$, we first solve the state-feedback problem to obtain the controller gain matrix F. Using a positive feedback convention, define

$$v = u - Fx. \tag{2.12}$$

With this feedback, the closed-loop system equations are

$$\dot{x} = (A + BF)x + Bv \tag{2.13}$$

$$y = (C + DF)x + Dv. \tag{2.14}$$

Solving for X(s) in (2.13) and then substituting it into (2.12) and (2.14) gives

$$u = \left[F(sI - A - BF)^{-1}B + I\right]v \tag{2.15}$$

$$\triangleq M\,v$$

and

$$y = \left[(C+DF)(sI - A - BF)^{-1}B + D\right]v \tag{2.16}$$

$$\triangleq N\,v$$

$$= NM^{-1}\,u. \tag{2.17}$$

Hence, the transfer function matrix G(s) is represented as

$$G(s) = N(s)\,M^{-1}(s) \tag{2.18}$$

where N and M are both stable because the regulator poles $\lambda_i(sI - A - BF)$ are gauranteed to be stable.

To obtain the left fractional representation, we solve the observer problem for the observer gain matrix H. Consider the observer equations

$$\dot{\zeta} = (A+HC)\zeta - Hy + (B+HD)u \tag{2.19}$$

$$\eta = C\zeta + Du - y. \tag{2.20}$$

Solving for $\zeta(s)$ in (2.19) and then substituting it into (2.20) yields

$$\eta = -\left[C(sI - A - HC)^{-1}H + I\right]y + \left[C(sI - A - HC)^{-1}(B+HD) + D\right]u$$

$$\triangleq -\tilde{M}y + \tilde{N}u. \tag{2.21}$$

By setting the error η to zero, we obtain

$$y = \tilde{M}^{-1}\,\tilde{N}\,u \tag{2.22}$$

which gives

$$G(s) = \tilde{M}^{-1}(s)\,\tilde{N}(s)\,. \tag{2.23}$$

$\tilde{M}$ and $\tilde{N}$ are both stable because the observer poles $\lambda_i(\,sI-A-HC\,)$ are gauranteed to be stable.

2. Internal Stability : Consider the block diagram shown in Figure 2 where P(s) is partitioned as

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}. \tag{2.24}$$

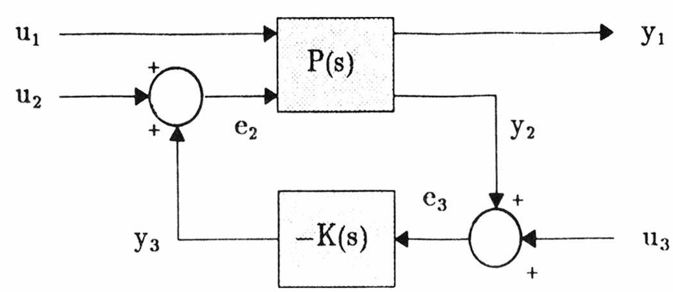

Figure 2 Block diagram for internal stability investigation

Assuming that P(s) is detectable, we say that the linear fractional feedback system is internally stable, or alternatively, K(s) stabilizes P(s), if all nine transfer functions from u_1, u_2 and u_3 to y_1, e_2 and e_3 are asymptotically stable. Similarly, if $P_{22}(s)$ is detectable, then K(s) stabilizes $P_{22}(s)$ if all four transfer functions from u_2 and u_3 to e_2 and e_3 are asymptotically stable. The next important lemma is taken from Francis (1987).

Lemma 2.1

(a) A stabilizable and detectable $P_{22}(s)$ implies that P(s) is stabilizable.

(b) If P(s) is stabilizable by linear fractional feedback, then K(s) stabilizes P(s) if and only if K(s) stabilizes $P_{22}(s)$.

$\triangle$

The following theorem by Hung (1989b) provides a necessary and sufficient condition for K(s) to stabilize P(s).

<u>Theorem 2.1</u>

Let

$$Q(s) \triangleq K(s)\left[\, I + P_{22}(s)\, K(s) \,\right]^{-1}. \tag{2.25}$$

With the assumptions that $P_{22}(s)$ is stabilizable and detectable and $P_{11}(s)$ is controllable and observable, then K(s) stabilizes P(s) if and only if the transfer functions

$$Q \;,\; P_{12}Q \;,\; Q\,P_{21} \;\text{ and }\; P_{11} - P_{12}QP_{21} \tag{2.26}$$

are asymptotically stable.

$\triangle$

3. Interpolation In H_∞ : An interpolation problem (IP) with an H_∞ criterion was first considered by Hung (1988). The problem is stated as follows :

<u>Problem Statement (H_∞ Interpolation)</u>

Given

$$\bar{N}(s) = \begin{bmatrix} \bar{N}_1(s) \\ \bar{N}_2(s) \end{bmatrix} \in I_- \qquad \text{and} \qquad \bar{M}(s) = \begin{bmatrix} \bar{M}_1(s) \\ \bar{M}_2(s) \end{bmatrix} \in H_- \tag{2.27}$$

with a combined realization

$$\begin{bmatrix} \bar{N}_1(s) & \bar{M}_1(s) \\ \bar{N}_2(s) & \bar{M}_2(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right], \tag{2.28}$$

find $E(s) \in H_\infty$ such that

$$\|E(s)\|_\infty \leq \gamma \quad , \quad \text{for some given positive scalar } \gamma \tag{2.29}$$

subject to

$$\bar{N}_1(s)\,E(s) = \bar{M}_1(s) \tag{2.30}$$

and

$$\left[\,\bar{N}_2(s)\,E(s)\,\right]_- = \bar{M}_2(s)\ . \tag{2.31}$$

Δ

The following theorem by Hung (1989a) provides a complete state-space solution to the above problem.

Theorem 2.2

In the notations given in the problem statement, suppose the realization for $\bar{N}(s)$ is minimal and balanced. Let $\gamma \geq \|\,\bar{M}_1(s)\,\|_\infty$ and

$$R = \gamma^2 I - D_{12}^T D_{12}. \tag{2.32}$$

Let $\hat{X} \geq 0$ be the unique destabilizing solution to the ARE

$$(A+B_2R^{-1}D_{12}^TC_1)\hat{X}+\hat{X}(A+B_2R^{-1}D_{12}^TC_1)^T-B_2R^{-1}B_2^T-\hat{X}C_1^T(I+D_{12}R^{-1}D_{12}^T)C_1\hat{X}=0 \tag{2.33}$$

such that

$$\lambda\,[\,(\,A+B_2R^{-1}D_{12}^TC_1\,) - \hat{X}\,C_1^T(\,I + D_{12}R^{-1}D_{12}^T)\,C_1\,] \subset \bar{\mathbb{C}}_+\ . \tag{2.34}$$

Also let

$$C_3 = R^{-\frac{1}{2}}(\,B_2^T - D_{12}^TC_1\hat{X}\,)\ . \tag{2.35}$$

Then

(a) Existence of solution : The IP has a solution iff $(\,I-\hat{X}\,) \geq 0$.

(b) Optimality : γ is the infinum iff one or both of the following conditions are satisfied

 (i) $(\,I-\hat{X}\,)$ is rank deficient

 (ii) $\gamma = \|\,\bar{M}_1(s)\,\|_\infty$.

(c) Characterization of optimal and sub-optimal solutions :

 (i) Suppose $(\,I-\hat{X}\,) \geq 0$ has a rank defect r>0. By applying a basis change to the realization (2.28), $(I-\hat{X})$ can be put in the form

$$I - \hat{X} = \begin{bmatrix} \hat{P} & 0 \\ 0 & 0_r \end{bmatrix} \tag{2.36}$$

where $\hat{P} = \hat{P}^T > 0$. Let the following matrices be partitioned conformally with (2.36) :

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B_1 = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix}, \tag{2.37}$$

$$C_1 = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}, \quad C_2 = \begin{bmatrix} C_{21} & C_{22} \end{bmatrix}, \quad \hat{C}_3 = \begin{bmatrix} \hat{C}_{31} & \hat{C}_{32} \end{bmatrix}.$$

Then, the IP is solved by

$$E(s) = \Phi (H(s) , U(s)) = H_{11} + H_{12}U (I-H_{22}U)^{-1}H_{21} \tag{2.38}$$

where

$$H(s) = \left[\begin{array}{c|cc} -A_{11}^T - C_{21}^T \mathcal{P} & B_{12} - C_{11}^T D_{12} & -C_{21}^T \\ \hline B_{11}^T + D_{21}^T \mathcal{P} & D_{11}^T D_{12} & D_{21}^T \\ \hat{C}_{31}\hat{P}^{-1} & -R^{\frac{1}{2}} & 0 \end{array} \right] \tag{2.39}$$

in which $\mathcal{P} = C_{21}(\hat{P}^{-1} - I)$ and $U(s) \in S_+$ is any sub-inner transfer function satisfying

$$C_{22}^T \, U(s) + \hat{C}_{32}^T = 0 \, . \tag{2.40}$$

(ii) If $(I-\hat{X}) > 0$ has full rank, the solution given by (2.38) and (2.39) remains valid with $\hat{P} = (I-\hat{X})$, $A_{11} = A$, $B_{11} = B_1$, etc. . The condition (2.40) is discarded in this case. Furthermore, if γ is sub-optimal, then all solutions to IP can be characterized in this way.

(d) McMillan degree bound : The solution E(s) to the IP satisfies

$$\deg (E(s)) \leq \deg (\overline{N}(s)) - r + \deg (U(s)) \, . \tag{2.41}$$

$\triangle$

4. Q-Parameterization : We first show a simple way to obtain a stabilizing controller for a given system. Consider the well-known closed-loop system equations with an observer-based state-feedback controller represented by

$$\dot{x} = (A + BF) x - H\eta \tag{2.42}$$

$$y = (C + DF) x + \eta \, . \tag{2.43}$$

Solving (2.42) for X(s) and substitution in (2.43) yields

$$y(s) = \left[-(C+DF) (sI-A-BF)^{-1}H + I \right] \eta \tag{2.44}$$

$$\triangleq V(s)\cdot\eta(s) \, .$$

From u=Fx, we obtain

$$u(s) = \left[-F (sI-A-BF)^{-1}H \right] \eta \tag{2.45}$$

$$\triangleq U(s)\cdot\eta(s) \, .$$

Hence,

$$u(s) = U(s) V^{-1}(s) y(s) \tag{2.46}$$

which gives

$$K(s) = U(s) V^{-1}(s) \tag{2.47}$$

where U and V are stable by their definitions. By a parallel procedure, we can derive

$$\tilde{V}(s) = -F (sI-A-HC)^{-1}(B+HD) + I \tag{2.48}$$

$$\tilde{U}(s) = -F (sI-A-HC)^{-1}H \tag{2.49}$$

and

$$K(s) = \tilde{V}^{-1}(s) \tilde{U}(s) \, . \tag{2.50}$$

We now give an important theorem from Maciejowski (1989) that parameterizes all stabilizing controllers with a stable parameter Q(s).

Theorem 2.3

(a) If $K_0 = U_0V_0^{-1} = \tilde{V}_0^{-1}\tilde{U}_0$ is a stabilizing controller for $G = MN^{-1} = \tilde{M}^{-1}\tilde{N}$, then $K = UV^{-1} = \tilde{V}^{-1}\tilde{U}$ is also a stabilizing controller for G where

$$U = U_0 + MQ\,, \qquad V = V_0 + NQ \tag{2.51}$$

$$\tilde{U} = \tilde{U}_0 + Q\tilde{M}\,, \qquad \tilde{V} = \tilde{V}_0 + Q\tilde{N} \tag{2.52}$$

for any $Q \in H_\infty$ (realizable and stable) .

(b) Any stabilizing controller has fractional representations in the form of (2.51) and (2.52).

$\triangle$

The above theorem indicates that a family of stabilizing controller could be easily generated once we have found a particular stabilizing controller. A general expression for the controller's transfer function K(s) is derived by Maciejowski (1989) starting from $K = (U_0+MQ)(V_0+NQ)^{-1}$, and is given by

$$K = K_0 + \tilde{V}_0^{-1}Q\,(I+V_0^{-1}NQ)^{-1}V_0^{-1}\,. \tag{2.53}$$

In block diagram representation, we can separate the parameter Q(s) from the controller K(s) and denote the fixed portion of K(s) by J(s) as illustrated in Figure 3.

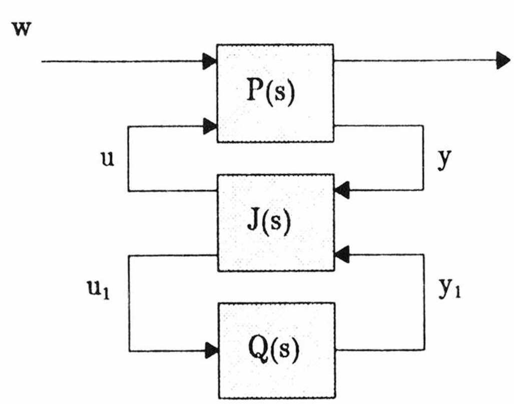

Figure 3 Separation of Q from K

The block denoted by P(s) in Figure 3 is known as the generalized plant and the significance of P(s) will be discussed in the next chapter.

5. Model-Matching Problem : Refering to Figure 3, we may incorporate the block J into P and denote the resulting block as T(s) as shown in Figure 4.

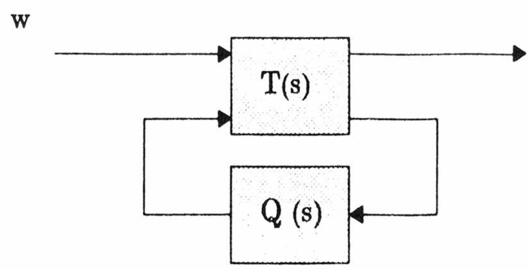

Figure 4 The block T(s) in model matching

If T(s) is partitioned as

$$T(s) = \begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & T_{22}(s) \end{bmatrix}, \tag{2.54}$$

then the transfer matrix from w to z is given by

$$z = \left[\, T_{11} + T_{12}Q\,(\,I - T_{22}Q\,)^{-1}T_{21}\,\right] w\ . \tag{2.55}$$

It is not difficult to derive the state-space realization for T(s) and it is obtained as

$$T_{11} = \left(\begin{bmatrix} A+B_2F & -B_2F \\ 0 & A+HC_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_1+HD_{21} \end{bmatrix}, \begin{bmatrix} C_1+D_{12}F & -D_{12}F \end{bmatrix}, \begin{bmatrix} D_{11} \end{bmatrix} \right) \tag{2.56}$$

$$T_{12} = \left(\begin{bmatrix} A+B_2F \end{bmatrix}, \begin{bmatrix} B_2 \end{bmatrix}, \begin{bmatrix} C_1+D_{12}F \end{bmatrix}, \begin{bmatrix} D_{12} \end{bmatrix} \right) \tag{2.57}$$

$$T_{21} = \left(\begin{bmatrix} A+HC_2 \end{bmatrix}, \begin{bmatrix} HD_{21}+B_1 \end{bmatrix}, \begin{bmatrix} C_2 \end{bmatrix}, \begin{bmatrix} D_{21} \end{bmatrix} \right) \tag{2.58}$$

$$T_{22} = 0\ . \tag{2.59}$$

In view of (2.59), equation (2.55) may be written as

$$z = \left[\, T_{11} + T_{12}Q\, T_{21}\,\right] w\ . \tag{2.60}$$

Since w represents the disturbance signal, we would like to equate its transfer function to zero. Henceforth, we are trying to determine a stable Q(s) so that the transfer matrix $-T_{12}Q\,T_{21}$ is best matched with T_{11}. This explains the name model-matching problem.

6. Loop Shifting : In order to facilitate the discussion in Section 4 of Chapter IV, we shall now present a series of operations known as loop shifting which we use to transform the given system to a particular form.

Given a plant $P(s) = (A, B, C, D)$ having a general D matrix in partition form as

$$D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}, \tag{2.61}$$

we wish to perform some operations on $P(s)$ so that the transformed plant $\hat{P}(s)$ will have a new D matrix having the form

$$\hat{D} = \begin{bmatrix} 0 & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ \begin{bmatrix} 0 & I \end{bmatrix} & 0 \end{bmatrix}. \tag{2.62}$$

This problem has been solved by Safonov et al. (1988,1989) and the following algorithm is taken from Dailey (1990).

Algorithm 2.1

Given a real constant $\gamma > 0$.

Step 1 : Use the singular value decomposition (SVD) to factor D_{12} and D_{21} :

$$D_{12} = U_1 \begin{bmatrix} 0 \\ \Sigma_1 \end{bmatrix} V_1^T \quad , \quad D_{21} = U_2 \begin{bmatrix} 0 & \Sigma_2 \end{bmatrix} V_2^T .$$

Step 2 : Scale D_{11} and partition it into a block 2x2 matrix where D_{1122} has the same dimension as D_{22}^T :

$$U_1^T D_{11} V_2 = \begin{bmatrix} D_{1111} & D_{1112} \\ D_{1121} & D_{1122} \end{bmatrix}.$$

Step 3 : Let

$$K_\infty = -\left[\; D_{1122} + D_{1121}(\, \gamma^2 I - D_{1111}^T D_{1111} \,)^{-1} D_{1111}^T D_{1112} \; \right].$$

Step 4 : Let M and the transformation matrix Θ be given by

$$M = U_1^T D_{11} V_2 + \begin{bmatrix} 0 & 0 \\ 0 & K_\infty \end{bmatrix} = \begin{bmatrix} D_{1111} & D_{1112} \\ D_{1121} & D_{1122}+K_\infty \end{bmatrix}$$

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \begin{bmatrix} -M & \sqrt{(I-\gamma^{-2}M M^T)} \\ \sqrt{(I-\gamma^{-2}M^T M)} & \gamma^{-2} M^T \end{bmatrix}.$$

Calculate

$$\hat{D}_{12} = (I - \gamma^{-2} M M^T)^{-\frac{1}{2}} \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$\hat{D}_{21} = \begin{bmatrix} 0 & I \end{bmatrix} (I - \gamma^{-2} M^T M)^{-\frac{1}{2}}$$

$$\hat{D}_{22} = \begin{bmatrix} 0 & I \end{bmatrix} \gamma^{-2} M^T (I - \gamma^{-2} M M^T)^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Step 5 : Use the SVD to factor $\hat{D}_{12}$ and $\hat{D}_{21}$:

$$\hat{D}_{12} = U_3 \begin{bmatrix} 0 \\ \Sigma_3 \end{bmatrix} V_3^T$$

$$\hat{D}_{21} = U_4 \begin{bmatrix} 0 & \Sigma_4 \end{bmatrix} V_4^T .$$

At this point, all the required matrices have been computed. The operations on P to achieve the desired form is depicted in Figure 5. We may group together the transformations above and below P into two matrices T_1 and T_2, respectively, which are given by

$$T_1 = \begin{bmatrix} -U_3^T M V_4 & U_3^T (I-\gamma^{-2} M M^T)^{\frac{1}{2}} U_1^T \\ V_2 (I-\gamma^{-2} M^T M)^{\frac{1}{2}} V_4 & \gamma^{-2} V_2 M^T U_1^T \end{bmatrix}$$

$$T_2 = \begin{bmatrix} T_{211} & T_{212} \\ T_{221} & T_{222} \end{bmatrix}$$

where

$$T_{211} = V_1 \Sigma_1^{-1} K_\infty \Sigma_2^{-1} U_2^T (I-L_1)^{-1}$$

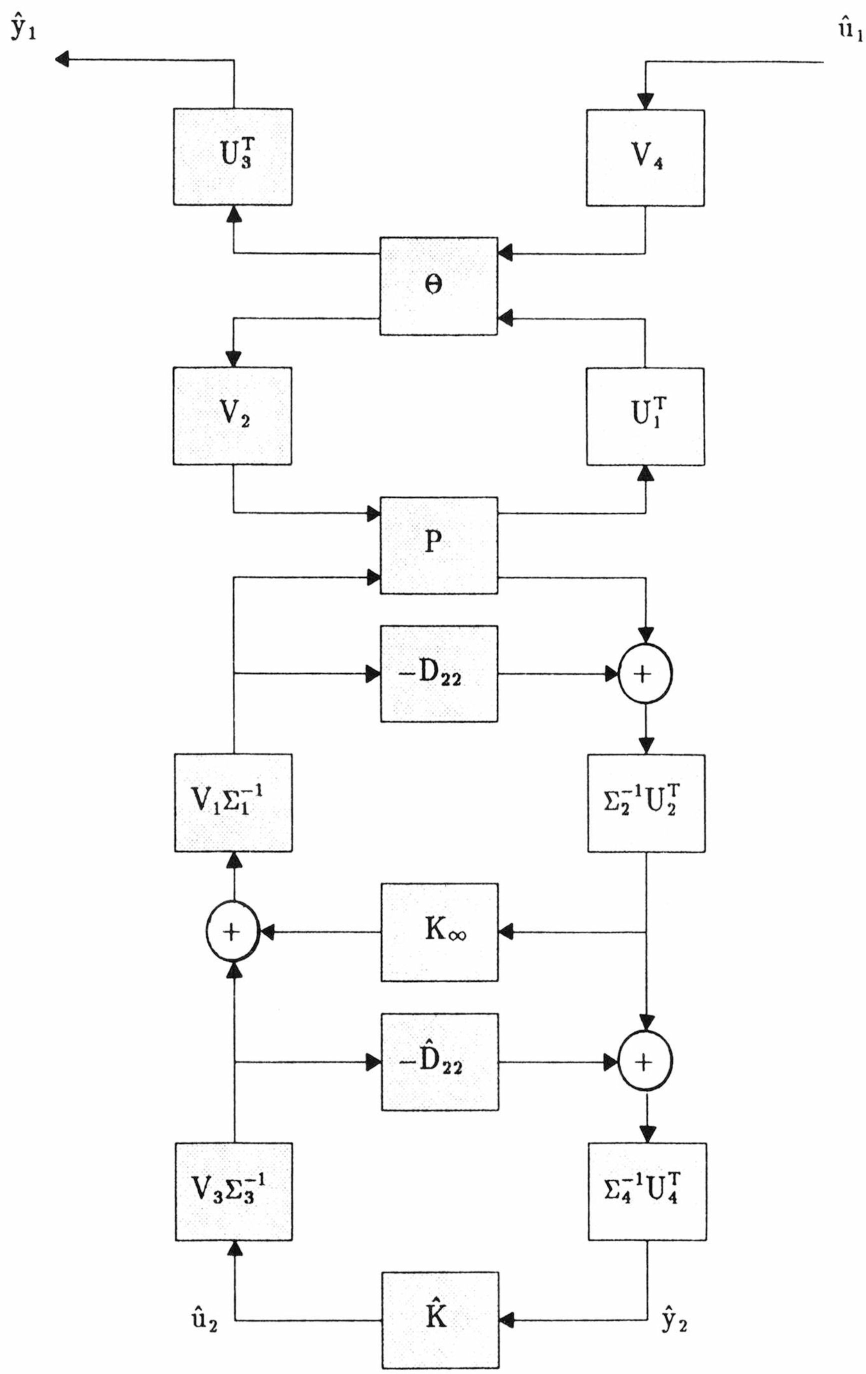

Figure 5 Loop shifting operation for H_∞

$$T_{212} = V_1 \Sigma_1^{-1} (I - L_2)^{-1} V_3 \Sigma_3^{-1}$$

$$T_{222} = - \Sigma_4^{-1} U_4^T \left[\Sigma_2^{-1} U_2^T D_{22} V_1 \Sigma_1^{-1} (I - L_2)^{-1} - \hat{D}_{22} \right] V_3 \Sigma_3^{-1}$$

$$L_1 = - D_{22} V_1 \Sigma_1^{-1} K_\infty \Sigma_2^{-1} U_2^T$$

$$L_2 = - K_\infty \Sigma_2^{-1} U_2^T D_{22} V_1 \Sigma_1^{-1} .$$

$\triangle$

After obtaining a H_∞ controller $\hat{K}$ for $\hat{P}$, it must be inverse-transformed before being applied to the original system P using the formula

$$K = T_{211} + T_{212} \hat{K} (I - T_{222} \hat{K})^{-1} T_{221} . \tag{2.63}$$

Notice that the transformation T_1 need not be inverse-transformed.

7. Conjugation : This section provides some basic definitions as well as some handy tools for obtaining a J-lossless conjugator. These are the important tools that were introduced in Kawatani and Kimura (1989), Kimura (1988, 1989) and Kimura and Kawatani (1988) for solving the H_∞ control problem.

Definition 2.1

A matrix $\Theta(s)$ of size $(m+r)\times(m+r)$ is said to be J-unitary if

$$\tilde{\Theta}(s)\, J\, \Theta(s) = J \tag{2.64}$$

holds for every s, where

$$J \triangleq \begin{bmatrix} I_m & 0 \\ 0 & -I_r \end{bmatrix} . \tag{2.65}$$

$\triangle$

Definition 2.2

A J-unitary matrix $\Theta(s)$ is said to be J-lossless if

$$\Theta^*(s)\, J\, \Theta(s) \leq J \qquad \text{for } \Re\{s\} \geq 0 \tag{2.66}$$

where $\Theta^*(s) = \Theta^T(\overline{s})$.

Δ

The following lemma gives the state-space characterization of J-unitary and J-lossless matrices.

Lemma 2.2

Let $\Theta(s) = (A, B, C, D)$ be a minimal realization. $\Theta(s)$ is J-unitary if and onlf if the relations

$$A^TP + PA + C^TJC = 0 \tag{2.67a}$$

$$D^TJC + B^TP = 0 \tag{2.67b}$$

$$D^TJD = J \tag{2.67c}$$

hold for a symmetric matrx P. It is J-lossless if and only if the above P is positive definite.

Δ

Conjugation is an operation that replaces the poles of a given system by their conjugates. In general, this is done by multiplying the system's transfer function by an all-pass function. However, it can also be performed in a state-space setting.

Let us consider a minimal realization of $G(s) = (A, B, C, D)$ and we seek a transfer function $V(s)$ such that

$$G(s)\, V(s) = (-A^T, *, *, *) \tag{2.68}$$

where $*$ denotes matrices whose form is irrelevant. Such a V always exists if G has no pure imaginary eigenvalues. V is said to be a conjugator of G and it is not unique. The following lemma furnishes us with a convenient tool for determining a conjugator for a given system.

Lemma 2.3

Let $G(s) = (A, B, C, D)$ be a minimal realization for a given system. A conjugator of G(s) is given by

$$V(s) = \left[\begin{array}{c|c} -A^T & P^{-1}BD_c \\ \hline -B^T & D_c \end{array}\right] \tag{2.69}$$

where P is the solution of the Lyapunov equation

$$AP + PA^T = BB^T \tag{2.70}$$

and D_c is any constant matrix.

△

The conjugation by a J-lossless conjugator is known as J-lossless conjugation and is the main operation used in one of the solution techniques for H_∞ control problem presented in Chapter IV. The following important lemma due to Kimura (1988) provides a state-space characterization of a J-lossless conjugator for a given system.

<u>Lemma 2.4</u>

Let the controllable pair (A, B) have no eigenvalue on the imaginary axis. Then a J-lossless conjugator of (A, B) exists iff the equation

$$AP + PA^T = BJB^T \tag{2.71}$$

has a positive definite solution, where J is given by (2.65). In that case, a J-lossless conjugator of (A, B) is given by

$$\Theta(s) = \left[\begin{array}{c|c} -A^T & P^{-1}BD_c \\ \hline -JB^T & D_c \end{array}\right] \tag{2.72}$$

where D_c is any constant J-unitary matrix.

△

III. H_∞ OPTIMAL CONTROL – PROBLEM FORMULATION

In this chapter, we first establish the H_∞ control problem statement in Section A before providing some H_∞ problem formulation examples in Section B.

A. STATEMENT OF PROBLEM

The basic block diagram used throughout the H_∞ control literature is shown below.

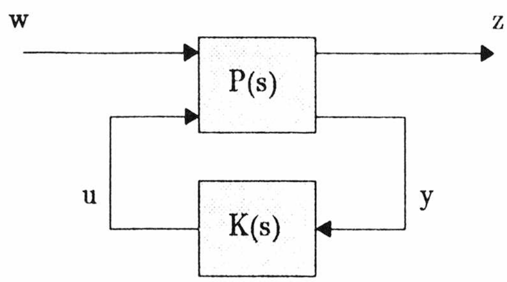

Figure 6 Basic block diagram in H_∞ control

The generalized plant P(s) consists of the nominal plant model and some weighting matrices introduced into the system to meet our design specifications. The signal w is a vector containing all external inputs to the system including command and disturbances. The output z is an error vector, u is the control vector while y is the measured output vector. Throughout this work, only linear time-invariant finite-dimensional systems will be considered.

Let the matrix P(s) be partitioned as

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}. \tag{3.1}$$

Direct expansion yields

$$z = P_{11}w + P_{12}u \tag{3.2}$$

$$y = P_{21}w + P_{22}u \,. \tag{3.3}$$

Using the relationship u=Ky, we obtain

$$y = (I - P_{22} K)^{-1} P_{21} w \tag{3.4}$$

$$z = \left[P_{11} + P_{12} K (I - P_{22}K)^{-1}P_{21} \right] w \tag{3.5}$$

$$= \Phi (P , K) w \,.$$

Since z is the error signal, we would always want to minimize $\Phi (P , K)$. Before defining the H_∞ control problem, it is necessary to give the definition of the infinity norm. The infinity norm of a given transfer matrix G(s) is defined as

$$\| G(s) \|_\infty \triangleq \sup_\omega \overline{\sigma} \, [G(j\omega)] \tag{3.6}$$

where 'sup' stands for 'supremium' and $\overline{\sigma} \, [\cdot]$ denotes the maximum singular value.

H_∞ Problem Statement

Given a generalized plant P(s) and a real constant $\gamma > 0$, find all stabilizing controllers K(s) such that

$$\|\Phi (P , K) \|_\infty < \gamma \,. \tag{3.7}$$

$\triangle$

B. EXAMPLES OF H_∞ CONTROL PROBLEM

1. Sensitivity Minimization : If we assume the disturbance d to enter the closed-loop system at the plant's output y, then it is well known that the sensitivity function S is defined as the transfer function from d to y as given below :

$$S \triangleq (I + GK)^{-1} \,. \tag{3.8}$$

For good disturbance rejection, the sensitivity function should be minimized over the frequency range where disturbances are significant, usually the low frequency range. Let $W(j\omega)$ be the weighting function such that

$$| W(j\omega) | \approx 1 \quad \text{for} \quad 0 \leq \omega \leq \omega_c , \quad \text{and}$$

$$| W(j\omega) | \ll 1 \quad \text{for} \quad \omega > \omega_c$$

where ω_c is the cutoff frequency for the disturbances. With the help of the Matrix Inversion Lemma given in the Apendix, S can be written as

$$S = I - GK (I + GK)^{-1} . \tag{3.9}$$

Hence, the problem can be posed as finding a stabilizing controller K such that

$$\| W [I - GK (I + GK)^{-1}] \|_{\infty} < \gamma . \tag{3.10}$$

Direct comparison of (3.5), (3.7) and (3.10) gives

$$P_{11} = W , \quad P_{12} = - WG , \quad P_{21} = I , \quad P_{22} = - G . \tag{3.11}$$

This implies that if we form the generalized plant P according to (3.11) and solve the H_∞ problem for K, then the closed-loop system will have minimum sensitivity to external disturbances in the frequency range $0 \leq \omega \leq \omega_c$.

2. Performance And Stability Requirements : It is readily verified that the closed-loop transfer function T from command input r to output y is given by

$$T = (I + GK)^{-1} GK . \tag{3.12}$$

With the definition (3.8), it is easy to derive the relationship

$$T = I - S . \tag{3.13}$$

In order to maintain closed-loop stability, the nominal loop gain is required to be small in the high frequency range where unstructured output multiplicative uncertainties is large, see Ridgely (1986). Hence, the problem of minimizing S and T in low and high frequency ranges respectively may be formulated as to find all stabilizing controllers K such that

$$\left\| \begin{matrix} W_1 S \\ W_2(I - S) \end{matrix} \right\|_{\infty} < \gamma \tag{3.14}$$

where $W_2(s)$ is complementary to $W_1(s)$. Following the same procedure as in the last section, it is obtained that the proper choices for the components of P are

$$P_{11} = \begin{bmatrix} W_1 \\ 0 \end{bmatrix}, \quad P_{12} = \begin{bmatrix} -W_1G \\ W_2G \end{bmatrix}, \quad P_{21} = I , \quad P_{22} = -G . \tag{3.15}$$

3. Linear Quadratic Regulator : We shall now show how the famous LQR problem can be posed in the framework of H_∞ control problem, which is called the H_2 problem because the L_∞-norm is replaced by the L_2-norm.

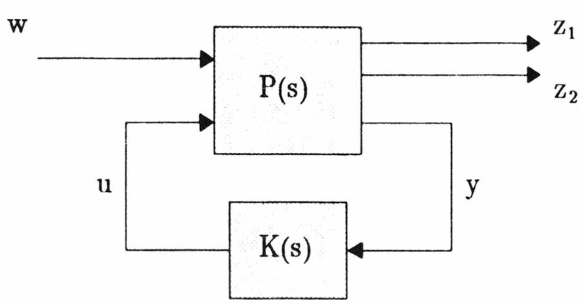

Figure 7 Block diagram for LQR control problem

Let us consider an additional error vector as shown in Figure 7. We shall assume P(s) to take on the general form as

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \\ C_3 & D_{31} & D_{32} \end{array}\right] . \tag{3.16}$$

The closed-loop output equation is given by

$$\begin{bmatrix} z_1 \\ z_2 \\ y \end{bmatrix} = \begin{bmatrix} C_1x + D_{11}w + D_{12}u \\ C_2x + D_{21}w + D_{22}u \\ C_3x + D_{31}w + D_{32}u \end{bmatrix} \tag{3.17}$$

and the error vector is defined as

$$z \triangleq \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} . \tag{3.18}$$

By definition, the L_2-norm of z is

$$\| z \|_2^2 = \int_0^{\infty} \{ z^T z \} \, dt . \tag{3.19}$$

By substituting (3.17) into (3.19) and after expanding and rearranging, we obtain

$$\| z \|_2^2 = \int_0^{\infty} \Bigg\{ x^T \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}^T \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x + 2\,x^T\, C_1^T \begin{bmatrix} D_{11} & D_{12} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$+ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^T \begin{bmatrix} D_{11}^T D_{11} & D_{11}^T D_{12} \\ D_{12}^T D_{11} & D_{12}^T D_{12} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + 2\,x^T\, C_2^T \begin{bmatrix} D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$+ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^T \begin{bmatrix} D_{21}^T D_{22} & D_{21}^T D_{22} \\ D_{22}^T D_{21} & D_{22}^T D_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Bigg\} \, dt . \tag{3.20}$$

If we make the following choices

$$C_1 = Q^{\frac{1}{2}}\, C \tag{3.21a}$$

$$C_2 = 0 \tag{3.21b}$$

$$D_{11} = D_{12} = D_{21} = 0 \tag{3.21c}$$

$$D_{22} = R^{\frac{1}{2}} \tag{3.21d}$$

where $(Q^{\frac{1}{2}})^T Q^{\frac{1}{2}} = Q = Q^T \geq 0$, $(R^{\frac{1}{2}})^T R^{\frac{1}{2}} = R = R^T > 0$ and define

$$\overline{z} = C_1\, x , \tag{3.22}$$

then we have

$$\| z \|_2^2 = \int_0^{\infty} \left\{ \overline{z}^T\, Q\, \overline{z} + u^T\, R\, u \right\} dt . \tag{3.23}$$

We immediately recognize that the right hand side of (3.23) is identical to the well known LQR performance index J.

From equation (3.17), we have

$$y = C_3 x + D_{31} w + D_{32} u \,. \tag{3.24}$$

We know that the LQR admits a state-feedback control law, hence y = x and the above equation gives

$$C_3 = I \tag{3.25a}$$

$$D_{31} = D_{32} = 0 \,. \tag{3.25b}$$

We can achieve any initial condition $x(0) = x_0$ by applying an impulse of magnitude x_0 to the input of the closed-loop system when it is at rest. Thus, we have

$$x(t) = \int_{-\infty}^{t} \left\{ A\, x(t) + B_1\, x_0 \delta(t) + B_2\, u(t) \right\} dt$$

$$X(0_+) = \int_{-\infty}^{0_+} A\, x(0_+)\, dt + \int_{-\infty}^{0_+} B_1\, x_0 \delta(0_+)\, dt + \int_{-\infty}^{0_+} B_2\, u(0_+)\, dt$$

$$= B_1\, x_0 \,.$$

The first and last integrals are equal to zero because we assume that the system is at rest prior to time t=0. If we choose

$$B_1 = I \tag{3.26}$$

then the initial condition is satisfied. With the choices (3.21), (3.25) and (3.26), we have

$$P(s) = \left[\begin{array}{c|cc} A & I & B \\ \hline \sqrt{Q}\, C & 0 & 0 \\ 0 & 0 & \sqrt{R} \\ I & 0 & 0 \end{array} \right] . \tag{3.27}$$

Notice that if we minimize $\|\Phi\,(P\,,K\,)\|_2$, then $\|z\|_2$ is also minimized, and so is J.

4. Kalman Filter : It is not surprising that the Kalman filter too can be posed in the framework of H_∞ control problem but again, it is called the H_2

problem for the same reason as LQR. The block diagram considered is shown in Figure 8.

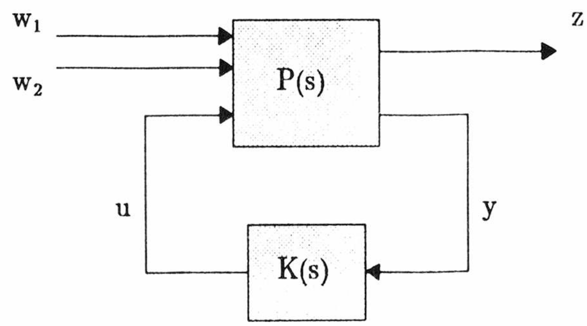

Figure 8 Block diagram for Kalman filter

The general form of P is

$$P(s) = \left[\begin{array}{c|ccc} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \end{array}\right] . \tag{3.28}$$

Here, the state vector that corresponds to P(s) is defined to be the error state vector, x_e. Hence, the error output equation is given by

$$z = C_1 x_e + D_{11} w_1 + D_{12} w_2 + D_{13} u . \tag{3.29}$$

Again, by the definition of the L_2-norm and with the choices

$$C_1 = I \tag{3.30a}$$

$$D_{11} = D_{12} = D_{13} = 0 , \tag{3.30b}$$

we obtain

$$\begin{aligned} \| z \|_2^2 &= \int_{-\infty}^{\infty} (x_e^T x_e) \, dt \\ &= E [x_e^T x_e] \\ &= J \end{aligned} \tag{3.31}$$

where E [·] denotes the expectation operator. The state equation is given by

$$\dot{x}_e = A x_e + B_1 w_1 + B_2 w_2 + B_3 u \tag{3.32}$$

and the control law is given by

$$u = H\,(\,C_2x_e + D_{21}w_1 + D_{22}w_2 + D_{23}u\,)\,. \tag{3.33}$$

Combining the above equations, we get

$$\dot{x}_e = (A+B_3HC_2)x_e + (B_1+B_3HD_{21})w_1 + (B_2+B_3HD_{22})w_2 + B_3HD_{23}u. \tag{3.34}$$

When we compare the error equation (3.34) with the error dynamic equation of the Kalman filter which is given by

$$\dot{x}_e = (\,A + KC\,)x_e + \Gamma\zeta + K\upsilon \tag{3.35}$$

where ζ and υ are the process and measurement noises respectively, then we obtain

$$B_1 = \Gamma\, W^{\frac{1}{2}}\ , \qquad B_2 = 0\ , \qquad B_3 = I\ , \qquad C_2 = C\ , \tag{3.36a}$$

$$D_{21} = 0\ , \qquad D_{22} = V^{\frac{1}{2}}\ , \qquad D_{23} = 0\ , \tag{3.36b}$$

where we have defined $w = W^{\frac{1}{2}}\,u_1$ and $\upsilon = V^{\frac{1}{2}}\,u_2$ in which $W = E\,[\,ww^T\,]$ and $V = E\,[\,\upsilon\upsilon^T\,]$. Thus the plant P is given by

$$P(s) = \left[\begin{array}{c|ccc} A & \Gamma\sqrt{W} & 0 & I \\ \hline I & 0 & 0 & 0 \\ C & 0 & \sqrt{V} & 0 \end{array}\right]. \tag{3.37}$$

Notice that when $\|\Phi\,(\,P\,,K\,)\|_2$ is minimized, $\|z\|_2$ and consequently J is also minimized.

IV. DESIGN OF H_∞ CONTROLLERS

The initial H_∞ control problem introduced by Zames (1981) was in an input-output setting which is still the most common case. However, in the case where all the states are available for feedback, it might be beneficial to feed back the states directly rather than estimating them. This is the reason behind the recent explorations of H_∞ state-feedback control problem by a number of authors, for instance, see Petersen (1987), Khargonekar et al. (1988), and Scherer (1990).

This chapter is divided into three sections. The first section concentrates on the solution techniques for the output feedback case while the second is devoted to the state-feedback case. A brief comparison of the methods together with some remarks are given in the last section.

A. OUTPUT FEEDBACK METHODS

1. Hankel Approximation : In section B.5 of Chapter II, we showed how the H_∞ control problem is equivalent to the model-matching problem. We shall now further transform this model-matching problem to yet another problem known as the Hankel Approximation problem or the Nehari Extension problem.

From equation (2.60), we have

$$\Phi(T, K) = T_{11} + T_{12} Q\, T_{21} \,. \tag{4.1}$$

The model-matching problem is to determine a stable Q(s) such that

$$\| T_{11} + T_{12} Q T_{21} \|_\infty$$

is minimized. We now make the assumption that T_{12} and T_{21} are square and all-pass which satisfy

$$T_{12}\tilde{T}_{12} = \tilde{T}_{21}T_{21} = I \, . \tag{4.2}$$

The dimensions of T_{12} and T_{21} are determined by the problem statement and by the number of system inputs and outputs. The matrices F and H in (2.57) and (2.58) may be selected such that the second assumption is satisfied. If X and Y are all-pass, then

$$\sigma(\, XAY \,) = \sigma(\, \tilde{A} \,) = \sigma(\, A \,) \, , \text{ where } \tilde{A}(s) = A^T(-s) \, , \tag{4.3}$$

for any A, where $\sigma(\cdot)$ denotes the singular value. Using (4.3), we have

$$\begin{aligned} \| T_{11}+T_{12}QT_{21} \|_\infty &= \| \tilde{T}_{11}+\tilde{T}_{21}\tilde{Q}\tilde{T}_{12} \|_\infty \\ &= \| \tilde{T}_{21}(\, T_{21}\tilde{T}_{11}T_{12} + \tilde{Q} \,)\tilde{T}_{12} \|_\infty \\ &= \| T_{21}\tilde{T}_{11}T_{12} + \tilde{Q} \|_\infty \\ &= \| G + \tilde{Q} \|_\infty \end{aligned} \tag{4.4}$$

where

$$G \triangleq T_{21}\tilde{T}_{11}T_{12} \, . \tag{4.5}$$

Thus, the model-matching problem has been converted into the following Hankel Approximation problem

$$\underset{Q \in H_\infty}{\text{minimize}} \| G + \tilde{Q} \|_\infty \tag{4.6}$$

where $G \in H_\infty$ is square.

The state-space realization of G is assumed to be $G = (\, A, B, C, D \,)$ and the controllability and observability gramians of G are denoted by P and R, respectively. The Hankel norm of G is defined as

$$\| G \|_H \triangleq \bar{\sigma}[\, (\, PR \,)^{\frac{1}{2}} \,] \, . \tag{4.7}$$

The next theorem which is due to Glover (1984) provides a lower bound for the attainable error for the Hankel Approximation problem.

Theorem 4.1

Given a rational transfer matrix $G \in H_\infty$ and any $Q \in H_\infty$,

$$\| G + \tilde{Q} \|_\infty \geq \| G \|_H . \tag{4.8}$$

Δ

The following algorithm and theorem developed by Glover (1984) provides a solution to the above stated problem.

Algorithm 4.1

Step 1 : Obtain a balanced realization for G and partition its gramian Σ as

$$\Sigma = \begin{bmatrix} \sigma_1 I & 0 \\ 0 & \Sigma_1 \end{bmatrix}$$

where $\Sigma_1 = \mathrm{diag}(\ \sigma_2, \cdots, \sigma_n\ )$ and $\sigma_1 > \sigma_2 \geq \cdots \geq \sigma_n > 0$. Then,

$$\| G \|_H = \sigma_1 .$$

Step 2 : Partition A, B and C conformally with Σ :

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} , \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} , \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} .$$

Define

$$\Gamma = \Sigma_1^2 - \sigma_1^2 I$$

and select an unitary U satisfying

$$B_1 = -\, C_1^T U .$$

Step 3 : Define

$$\hat{A} = \Gamma^{-1}(\ \sigma_1^2 A_{22}^T + \Sigma_1 A_{22} \Sigma_1 - \sigma_1 C_2^T U B_2^T\ )$$

$$\hat{B} = \Gamma^{-1}(\ \Sigma_1 B_2 + \sigma_1 C_2^T U\ )$$

$$\hat{C} = -\, C_2 \Sigma_1 - \sigma_1 U B_2^T$$

$$\hat{D} = -\, D + \sigma_1 U$$

and let Q_{opt} have the realization ($-\hat{A}^T, -\hat{C}^T, \hat{B}^T, \hat{D}^T$).

Δ

Theorem 4.2

Given a real rational transfer matrix $G \in H_\infty$ and let Q_{opt} be defined as in Algorithm 4.1. Then

(a) $Q_{opt} \in H_\infty$

(b) $\| G + \tilde{Q}_{opt} \|_\infty = \| G \|_H$

(c) $M \tilde{M} = \sigma_1^2 I$, where

$$M \triangleq G + \tilde{Q}_{opt} .$$

△

In many cases, it is sufficient to obtain a sub-optimal solution, namely Y_{sub} , such that

$$\| G + \tilde{Q}_{sub} \|_\infty \leq \delta \tag{4.9}$$

for a given real scalar $\delta > 0$. A sub-optimal solution exists if and only if $\delta \geq \| G \|_H$. Safonov et al. (1987) have modified the above algorithm for the sub-optimal case. This modified algorithm for solving the model-matching problem is given below :

Algorithm 4.2

Step 1 : Solve for P and R, the controllability and observability gramians respectively and calculate the Hankel norm of G. If $\| G \|_H > \delta$, then the solution does not exist.

Step 2 : Define

$$\Gamma_\delta = RP - \delta^2 I$$

$$\hat{A}_\delta = \Gamma_\delta^{-1}(\delta^2 A^T + RPA)$$

$$\hat{B}_\delta = \Gamma_\delta^{-1} RB$$

$$\hat{C}_\delta = - CP$$

$$\hat{D}_\delta = - D$$

and let Q_{sub} have the realization $(-\hat{A}_\delta^T, -\hat{C}_\delta^T, \hat{B}_\delta^T, \hat{D}_\delta^T)$.

△

Theorem 4.3

Given a real rational transfer matrix $G \in H_\infty$ and let Q_{sub} be defined as in Algorithm 4.2. Then,

(a) $Q_{sub} \in H_\infty$

(b) $\| G + \tilde{Q}_{sub} \|_\infty \leq \delta$.

$\triangle$

2. H_∞ Interpolation : In Chapter II, we have introduced the H_∞ interpolation problem together with its state-space solution. Also in Chapter II, we have shown how the H_∞ control problem can be transformed to the model-matching problem. The general model-matching problem as defined in equation (2.60) is known as the double-sided model-matching problem because the parameter Q(s) is multiplied on both sides by two transfer matrices. A more restrictive class of model-matching problem in which Q(s) is multiplied only on one side is called the one-sided model-matching problem. It is shown by Hung (1989a) that the one-sided model-matching problem can be converted into a H_∞ interpolation problem which is readily solved. The conversion procedure is quite straightforward and once it has been converted, Theorem 2.2 can be applied directly. The interested readers may refer to Hung (1989a) for the solution of this one-sided model-matching problem based on H_∞ interpolation theory.

In has been demonstrated in Hung (1989b) that the H_∞ optimal control problem could be transformed into two one-sided model-matching problems which can be solved one after another using the solution technique discussed above. A detailed derivation of such procedure which involves the use of several lemmas and lengthy state-space manipulations is presented in Hung (1989b) and will not be repeated here. Instead, we present a summary of those results as follows.

The generalized plant matrix P(s) in Figure 9 is assumed to have a minimal realization given by

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{4.10}$$

where

$$P_{ij}(s) = C_i\,(\,sI - A\,)^{-1}B_j + D_{ij} \in \mathbf{R}(s)^{p_i * m_j} \;, \qquad i, j = 1, 2\,. \tag{4.11}$$

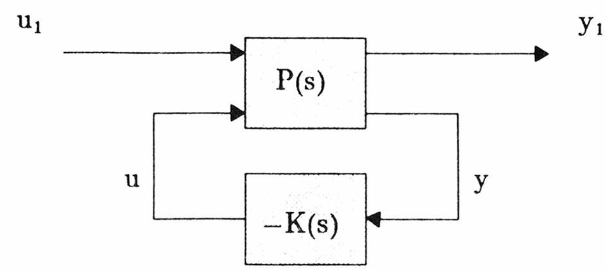

Figure 9 H_∞ control block diagram for interpolation theory

The following assumptions are made on the realization of P(s).

Assumption 4.1

(a) (A, B_2, C_2) is stabilizable and detectable.

(b) P_{12} and P_{21} are non-square with P_{12} 'tall' and P_{21} 'fat'.

(c) D_{12} and D_{21} have full column and row ranks respectively.

Assumption 4.2

(a) (A, B_1, C_1) is controllable and observable.

(b) D_{12} is column orthogonal, i. e. $D_{12}^T D_{12} = I$.

Assumption 4.1(b) confines our attention to the H_∞ control problem of the third kind. However, problems of the first and second kinds in which P_{12} and/or P_{21} is square, can be included easily by augmenting zero sub-blocks to the right of P_{11} and P_{21} and/or below P_{11} and P_{12}. Lemma 4.1 below shows how Assumption 4.2(b) could be satisfied.

<u>Lemma 4.1</u>

Let

$$K = L\tilde{K} + D \tag{4.12}$$

where $D \in \mathbf{R}^{m_2 * p_2}$ satisfies $\det(I + DD_{22}) \neq 0$ and $L \in \mathbf{R}^{m_2 * m_2}$ is non-singular. Then

$$\Phi (P , -K) = \Phi (\tilde{P} , -\tilde{K}) \tag{4.13}$$

where

$$\tilde{P}(s) = \left[\begin{array}{c|cc} A-B_2\mathfrak{X} & B_1-B_2\mathfrak{Y} & B_2\Delta L \\ \hline C_1-D_{12}\mathfrak{X} & D_{11}-D_{12}\mathfrak{Y} & D_{12}\Delta L \\ \delta\, C_2 & \delta\, D_{21} & D_{22}\Delta L \end{array} \right] \tag{4.14}$$

in which

$$\mathfrak{X} = \Delta D\, C_2 \ , \qquad \mathfrak{Y} = \Delta D\, D_{21} \ , \tag{4.15a}$$

$$\Delta = (I + DD_{22})^{-1} \qquad \text{and} \qquad \delta = (I + D_{22}D)^{-1} \ . \tag{4.15b}$$

There exist D and L such that ($\tilde{A}, \tilde{B}_1, \tilde{C}_1$) is minimal and $\tilde{D}_{12}$ is column orthogonal.

Δ

Note that the properties of Assumption 4.1 are invariant under the operation described in Lemma 4.1 and that the controller K(s) must be recovered from $\tilde{K}(s)$ obtained for $\tilde{P}(s)$ using the relationship (4.12).

For mathematical simplicity, the last row of (4.10) is pre-multiplied by a factor of L_γ that satisfies

$$(L_\gamma D_{21} R^{-\frac{1}{2}}) (L_\gamma D_{21} R^{-\frac{1}{2}})^T = I \tag{4.16}$$

where

$$R = \gamma^2 I - D_{11}^T D_{13} D_{13}^T D_{11} > 0 \tag{4.17}$$

and D_{13} is the orthogonal complement of D_{12}. We shall asumme that this scaling has already been done in (4.10) and will not bear the notation L_γ except keeping in mind that the reverse scaling L_γ^{-1} has to be done to the controller K at the end.

The matrices P and $\tilde{P}$ are the solutions of the algebraic Riccati equations

$$PZ + Z^T P - PWP + V = 0 \tag{4.18}$$

and

$$\tilde{P}\tilde{Z} + \tilde{Z}^T\tilde{P} - \tilde{P}\tilde{W}\tilde{P} + \tilde{V} = 0 \tag{4.19}$$

where

$$Z = -\left[A - B_2 D_{12}^T C_1 + (B_1 - B_2 D_{12}^T D_{11}) R^{-1} D_{11}^T D_{13} D_{13}^T C_1 \right]^T \tag{4.20}$$

$$W = C_1^T D_{13} (I + D_{13}^T D_{11} R^{-1} D_{11}^T D_{13}) D_{13}^T C_1 \tag{4.21}$$

$$V = B_2 B_2^T - (B_1 - B_2 D_{12}^T D_{11}) R^{-1} (B_1 - B_2 D_{12}^T D_{11})^T \tag{4.22}$$

$$\tilde{Z} = -\left[\tilde{A} - \tilde{B}_2 \tilde{D}_2^T \tilde{C} + (\tilde{B}_1 - \tilde{B}_2 \tilde{D}_2^T \tilde{D}_1) \tilde{R}^{-1} \tilde{D}_1^T \tilde{D}_3 \tilde{D}_3^T \tilde{C} \right]^T \tag{4.23}$$

$$\tilde{W} = \tilde{C}^T \tilde{D}_3 (I + \tilde{D}_3^T \tilde{D}_1 \tilde{R}^{-1} \tilde{D}_1^T \tilde{D}_3) \tilde{D}_3^T \tilde{C} \tag{4.24}$$

$$\tilde{V} = \tilde{B}_2 \tilde{B}_2^T - (\tilde{B}_1 - \tilde{B}_2 \tilde{D}_2^T \tilde{D}_1) \tilde{R}^{-1} (\tilde{B}_1 - \tilde{B}_2 \tilde{D}_2^T \tilde{D}_1)^T \tag{4.25}$$

$$\tilde{R} = I - \tilde{D}_1^T \tilde{D}_3 \tilde{D}_3^T \tilde{D}_1 \tag{4.26}$$

in which

$$\tilde{A} = A^T + S^T R^{-\frac{1}{2}} B_1^T \tag{4.27}$$

$$\tilde{B}_1 = S^T R^{-\frac{1}{2}} D_{11}^T D_{12} + (C_1^T D_{12} + P^{-1} B_2) \tag{4.28}$$

$$\tilde{B}_2 = -(C_2^T + S^T R^{-\frac{1}{2}} D_{21}^T) \tag{4.29}$$

$$\tilde{C} = - R^{-\frac{1}{2}} B_1^T \tag{4.30}$$

$$\tilde{D}_1 = - R^{-\frac{1}{2}} D_{11}^T D_{12} \tag{4.31}$$

$$\tilde{D}_2 = R^{-\frac{1}{2}} D_{21}^T \tag{4.32}$$

$$S = R^{-\frac{1}{2}}\left[(B_1 - B_2 D_{12}^T D_{11})^T P^{-1} + D_{11}^T D_{13} D_{13}^T C_1 \right] \tag{4.33}$$

and $\tilde{D}_3$ is the orthogonal complement of $\tilde{D}_2$. Then an orthogonal congruence transformation $\tilde{T}$ is to be determined such that

$$\tilde{T} (\tilde{Y}^{\frac{1}{2}} \tilde{P} \tilde{Y}^{\frac{1}{2}}) \tilde{T}^T = \begin{bmatrix} \hat{P} & 0 \\ 0 & 0_{\tilde{r}} \end{bmatrix} \tag{4.34}$$

where $\tilde{Y}$ is the solution of the ARE

$$(\tilde{A} - \tilde{B}_2 \tilde{D}_2^T \tilde{C})^T \tilde{Y} + \tilde{Y}(\tilde{A} - \tilde{B}_2 \tilde{D}_2^T \tilde{C}) - \tilde{Y} \tilde{B}_2 \tilde{B}_2^T \tilde{Y} + \tilde{C}^T \tilde{D}_3 \tilde{D}_3^T \tilde{C} = 0 \tag{4.35}$$

and

$$\tilde{r} = \mathrm{def} (\tilde{P}) , \tag{4.36}$$

where def($\cdot$) denotes the rank defect. A similarity transformation with $T = \tilde{T}\tilde{Y}^{\frac{1}{2}}$ is then applied to (4.27) - (4.30) before they are partitioned conformally with (4.34) as

$$T\tilde{A}T^{-1} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad T\tilde{B}_1 = \begin{bmatrix} \tilde{B}_{11} \\ \tilde{B}_{21} \end{bmatrix}, \quad T\tilde{B}_2 = \begin{bmatrix} \tilde{B}_{12} \\ \tilde{B}_{22} \end{bmatrix}, \tag{4.37a}$$

$$\tilde{C}T^{-1} = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix} . \tag{4.37b}$$

The following matrices are defined :

$$\tilde{S}_1 = \tilde{R}^{-\frac{1}{2}}\left[(\tilde{B}_{11} - \tilde{B}_{12} \tilde{D}_2^T \tilde{D}_1)^T \hat{P}^{-1} + \tilde{D}_1^T \tilde{D}_3 \tilde{D}_3^T \tilde{C}_1 \right] \tag{4.38}$$

$$\tilde{S}_2 = \tilde{R}^{-\frac{1}{2}} (\tilde{B}_{21} - \tilde{B}_{22} \tilde{D}_2^T \tilde{D}_1)^T \tag{4.39}$$

$$\overline{B} = - \tilde{S}_1^T \tilde{R}^{-\frac{1}{2}} \tilde{D}_1^T \tilde{D}_2 - (\tilde{C}_1^T \tilde{D}_2 + \hat{P}^{-1} \tilde{B}_{12}) \tag{4.40}$$

<u>Theorem 4.4</u>

Suppose we have chosen a γ large enough to permit a solution to the H_∞ control problem. Then the linear fractional feedback system of Figure 9 is internally stable with

$$\| \Phi (P(s) , -K(s)) \|_\infty \leq \gamma \tag{4.41}$$

and K(s) is given by

$$K(s) = \Psi(H(s), U(s)) = (H_{11} + UH_{21})^{-1}(H_{21} + UH_{22}) \tag{4.42}$$

where

$$H(s) = (A_h, B_h, C_h, D_h) \tag{4.43}$$

$$A_h = \left[\; \tilde{A}_{11}^T + \tilde{S}_1^T \tilde{R}^{-\frac{1}{2}} \tilde{B}_{11}^T + \bar{B}\,\tilde{B}_{12}^T \;\right]$$

$$B_h = \left[\; \tilde{S}_1^T \tilde{R}^{-\frac{1}{2}} - \bar{B}D_{22} + B_{12} \qquad \bar{B} \;\right]$$

$$C_h = \begin{bmatrix} \tilde{R}^{-\frac{1}{2}}(\tilde{B}_{11}^T - \tilde{D}_1^T \tilde{D}_2 \tilde{B}_{12}^T) \\ -\tilde{B}_{12}^T \end{bmatrix}$$

$$D_h = \begin{bmatrix} \tilde{R}^{-\frac{1}{2}}(I + \tilde{D}_1^T \tilde{D}_2 D_{22}) & -\tilde{R}^{-\frac{1}{2}} \tilde{D}_1^T \tilde{D}_2 \\ D_{22} & -I \end{bmatrix}$$

in which the partition

$$T^{-T} B_2 = \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} \tag{4.44}$$

has been used. $U(s) \in S_+$ is any sub-inner transfer function that must also satisfy

$$\tilde{B}_{22} U^T(s) - \tilde{S}_2^T = 0 \tag{4.45}$$

if $\tilde{r} > 0$.

$\triangle$

An iterative algorithm for the computation of a H_∞ optimal or sub-optimal controller is provided below.

Algorithm 4.3

Given P(s) with a minimal realization (4.10) that satisfies Assumption 4.1.

Step 1 : Perform the operation in Lemma 4.1 to ensure that Assumption 4.2 is satisfied.

Step 2 : Choose a value for γ and do the scaling described in (4.16).

Step 3 : Solve ARE (4.18). If no stabilizing solution exist, then set $\gamma_l = \gamma$, increase γ and return to Step 2, else (P>0) continue.

Step 4 : Solve ARE (4.19). If no stabilizing solution exist, then set $\gamma_l = \gamma$, increase γ and return to Step 2, else ($\tilde{P} \geq 0$) either decrease γ and return to Step 2 or continue (satisfied with the value of γ).

Step 5 : If $\tilde{r}$=def($\tilde{P}$)>0, then solve ARE (4.35), do the operations in (4.34) and (4.37) and choose a particular sub-inner solution U(s) satisfying (4.45), else choose any sub-inner transfer function U(s).

Step 6 : Determine K(s) using (4.42) - (4.44).

Step 7 : Invert the scaling of Step 2 and reverse the operation in Step 1.

$\triangle$

3. J-lossless Conjugation : Using the J-lossless conjugation approach to solve the H_∞ control problem in its model-matching form has been proposed by Kimura (1988) and later been extended by Kimura and Kawatani (1988). They have shown that the J-lossless conjugation is a powerful tool in dealing with the two-sided model-matching problem and have derived a new state-space formula for the H_∞ controller. A detailed derivation of the technique together with an algorithm for the calculation of a H_∞ controller is presented by Kawatani and Kimura (1989) and is summarized below.

The standard block diagram for H_∞ control problem as shown in Figure 10 is considered where the n^{th} order generalized plant P(s) has a realization

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right] \tag{4.46}$$

where $A \in \mathbf{R}^{m*n}$, $B_1 \in \mathbf{R}^{n*m_1}$, $B_2 \in \mathbf{R}^{n*m_2}$, $C_1 \in \mathbf{R}^{p_1*n}$, $C_2 \in \mathbf{R}^{p_2*n}$, $D_{11} \in \mathbf{R}^{p_1*m_1}$,

$D_{12} \in \mathbf{R}^{p_1 * m_2}$ and $D_{21} \in \mathbf{R}^{p_2 * m_1}$.

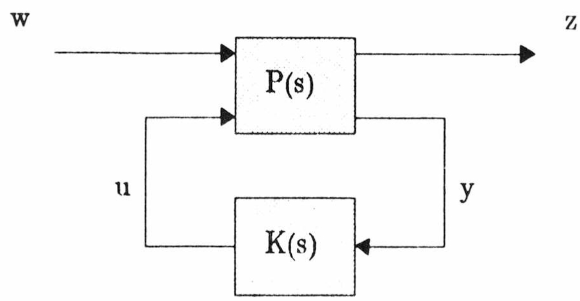

Figure 10 Block diagram for H_∞ control problem

With the implementation of the control law $u = Ky$, the closed-loop transfer function from w to z is given by

$$z = \Phi\, w \tag{4.47}$$

where $\Phi = T_{11} - T_{12} Q T_{21}$ and $T_{11} \in \mathbf{R}H_\infty^{m*r}$, $T_{12} \in \mathbf{R}H_\infty^{m*m}$, $T_{21} \in \mathbf{R}H_\infty^{r*r}$, $Q \in \mathbf{R}H_\infty^{m*r}$ and the state-space realization for T_{11}, T_{12} and T_{21} are given in (2.56) – (2.58). The problem is to determine a stable Q, if it exists, such that

$$\| \Phi \|_\infty < 1 \, . \tag{4.48}$$

The error is normalized to unity in (4.48) and the general case will be given later. We make the following assumptions.

<u>Assumption 4.3</u>

(a) (A, B_2, C_2) is stabilizable and detectable.

(b) Both T_{12}^{-1} and T_{21}^{-1} exist (both D_{12} and D_{21} are square and have full rank).

(c) T_{12}^{-1} has no unstable poles in common with T_{21}^{-1}.

Assumption 4.3(b) restricts our attention to the so-called model-matching problem of the first kind (where both T_{12} and T_{21} are square). This assumption is removed in Kimura and Kawatani (1988).

The following definitions are made :

$$A_f = A + B_2 F \quad , \qquad A_h = A + H C_2 \tag{4.49}$$

$$L_2 = B_2 D_{12}^{-1} \quad , \quad \hat{A}_2 = A - L_2 C_1 \tag{4.50}$$

$$L_3^T = D_{21}^{-1} C_2 \quad , \quad \hat{A}_3 = A - B_1 L_3^T \tag{4.51}$$

$$M_2 = L_2 D_{11} - B_1 \quad , \quad M_3^T = D_{11} L_3^T - C_1 \tag{4.52}$$

where F and H are given in the realizations of T_{ij}, $i, j = 1, 2$. Notice that no assumption on the stability of $\hat{A}_2$ and $\hat{A}_3$ was made and thus they may be decomposed into their stable portions, $\hat{A}_{2s}$ and $\hat{A}_{3s}$ with dimensions $n_{2s} x n_{2s}$ and $n_{3s} x n_{3s}$ respectively, and thier unstable portions $\hat{A}_{2u}$ and $\hat{A}_{3u}$ with dimensions $n_{2u} x n_{2u}$ and $n_{3u} x n_{3u}$ respectively. These decompositions were done by determining similarity transformation matrices S_2 and S_3 such that

$$S_2 \hat{A}_2 S_2^{-1} = \begin{bmatrix} \hat{A}_{2u} & 0 \\ 0 & \hat{A}_{2s} \end{bmatrix} \tag{4.53}$$

and

$$S_2 \hat{A}_3 S_3^{-1} = \begin{bmatrix} \hat{A}_{3u} & 0 \\ 0 & \hat{A}_{3u} \end{bmatrix} . \tag{4.54}$$

These transformation matrices were then partitioned conformally with (4.53) and (4.54) like so :

$$S_2 = \begin{bmatrix} S_{21} \\ S_{22} \end{bmatrix} \quad , \quad S_3 = \begin{bmatrix} S_{31} \\ S_{32} \end{bmatrix} \quad , \tag{4.55}$$

$$S_2^{-1} = \begin{bmatrix} \hat{S}_{21} & \hat{S}_{22} \end{bmatrix} \quad , \quad S_3^{-1} = \begin{bmatrix} \hat{S}_{31} & \hat{S}_{32} \end{bmatrix} . \tag{4.56}$$

A symmetric matrix P_0 is defined to be

$$P_0 = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \tag{4.57}$$

where

$$P_{12} = S_{21} \hat{S}_{31} \tag{4.58}$$

and P_{11} and P_{22} are the solutions to the Lyapunov equations

$$\hat{A}_{2u}P_{11} + P_{11}\hat{A}_{2u}^T = S_{21}(\,L_2L_2^T - M_2M_2^T\,)S_{21}^T \tag{4.59}$$

$$\hat{A}_{3u}^T P_{22} + P_{22}\hat{A}_{3u}^T = \hat{S}_{31}^T(\,L_3L_3^T - M_3M_3^T\,)\hat{S}_{31}\ . \tag{4.60}$$

The following matrices are constructed :

$$A_A = \begin{bmatrix} -\hat{A}_{2u}^T & 0 \\ S_{22}(\,M_2M_2^T - L_2L_2^T\,)S_{21}^T & \hat{A}_{2s} \end{bmatrix} + B_{A1}C_{A2} \tag{4.61a}$$

$$B_A = \begin{bmatrix} B_{A1}D_{21}^{-1} & B_{A2} \end{bmatrix} \tag{4.61b}$$

$$C_A = \begin{bmatrix} D_{12}^{-1}C_{A1} \\ C_{A2} \end{bmatrix} \tag{4.61c}$$

$$D_A = \begin{bmatrix} D_{12}^{-1}D_{11}D_{21}^{-1} & -D_{12}^{-1} \\ D_{21}^{-1} & 0 \end{bmatrix} \tag{4.61d}$$

where

$$\begin{bmatrix} B_{A1} & B_{A2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -S_{22}M_2 & S_{22}L_2 \end{bmatrix} + \begin{bmatrix} I_{n_{2u}} & 0 \\ 0 & -S_{22}\hat{S}_{31} \end{bmatrix} P_0^{-1} \begin{bmatrix} -S_{21}M_2 & S_{21}L_2 \\ -\hat{S}_{31}^T L_3 & \hat{S}_{31}^T M_3 \end{bmatrix} \tag{4.62a}$$

$$\begin{bmatrix} C_{A1} \\ C_{A2} \end{bmatrix} = \begin{bmatrix} L_2^T S_{21}^T - M_3^T \hat{S}_{21}P_{11} & -M_3^T \hat{S}_{22} \\ M_2^T S_{21}^T - L_3^T \hat{S}_{21}P_{11} & -L_3^T \hat{S}_{22} \end{bmatrix} . \tag{4.62b}$$

Theorem 4.5

Under Assumption 4.3, a stabilizing controller K(s) that achieves (4.48) exists if and only if the matrix P_0 in (4.57) is positive definite. In that case, the controller K(s) is represented as

$$K(s) = \Lambda_{11} + \Lambda_{12}\Psi\,(\,I - \Lambda_{22}\Psi\,)^{-1}\Lambda_{21} \quad , \qquad \Psi(s) \in BH_\infty^{m*r} \tag{4.63}$$

where $\Lambda(s) = (\,A_A, B_A, C_A, D_A\,)$ and the matrices A_A, B_A, C_A, D_A are evaluated by using (4.61) and (4.62).

△

For the central or maximum entropy controller which corresponds to the choice $\Psi(s)=0$, the resulting controller $K(s) = \Lambda_{11}(s)$ has the same order as the plant $P(s)$. Also note that $K(s)$ is independent of the stabilizing matrices F and H.

We have been concerning ourselves thus far with the consideration of the normalized error criterion (4.48). For the general case where we have

$$\| \Phi \|_\infty < \gamma , \tag{4.64}$$

all the discussions above are still valid provided the replacements

$$M_2 \rightarrow \gamma^{-1} M_2 \qquad \text{and} \qquad M_3 \rightarrow \gamma^{-1} M_3 \tag{4.65}$$

are made.

We now present a simple algorithm for the computation of a H_∞ controller.

Algorithm 4.4

Step 1 : For a given plant $P(s)$ as in (4.46), calculate L_2, L_3, $\hat{A}_2$, $\hat{A}_3$, M_2 and M_3.

Step 2 : Determine S_2, S_2^{-1}, S_3 and S_3^{-1} in partitioned form as in (4.55) and (4.56) and transform $\hat{A}_2$ and $\hat{A}_3$ into modal form given in (4.53) and (4.54) respectively.

Step 3 : Solve the Lyapunov equations (4.59) and (4.60) for P_{11} and P_{22} and compute P_{12} from (4.58) and check whether P_0 in (4.57) is positive definite.

Step 4 : If $P_0 > 0$, then the controller $K(s)$ is obtained from (4.63), (4.61) and (4.62).

$\triangle$

A software program has been developed in Matrix_X for the implementation of the above algorithm. An example shows that the results produced by the above algorithm are satisfactory.

4. General Distance Problem : The earliest state-space solution to the H_∞ control problem was presented by Doyle (1984) who transformed the model-matching problem into a 2x2 block general distance problem and then reduced it to the Nehari problem before solving it directly. Unfortunately, this approach involves solving several high dimension Riccati equations and the resulting high order controller is very complicated. This approach was modified in Doyle et al. (1989) which results in a controller having a dimension identical to the plant's order. The results of this method are summarized below.

We again consider the standard block diagram for the H_∞ control problem shown in Figure 10. The generalized plant P(s) is taken to be of the form

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \quad P_{ij} \in \mathbf{R}(s)^{p_i * m_j}, \quad i, j = 1, 2, \tag{4.66}$$

and the problem is to find K(s) such that

$$\| \Phi \|_\infty < \gamma . \tag{4.67}$$

Assumption 4.4

(i) (A, B_1, C_1) is stabilizable and detectable.

(ii) (A, B_2, C_2) is stabilizable and detectable.

(iii) $D_{11} = D_{22} = 0$.

(iv) $D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$.

(v) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$.

X_∞ and Y_∞ are defined to be the solutions to the AREs

$$A^T X_\infty + X_\infty A + X_\infty(\gamma^{-2}B_1B_1^T - B_2B_2^T)X_\infty + C_1^T C_1 = 0 \tag{4.68}$$

$$A Y_\infty + Y_\infty A^T + Y_\infty(\gamma^{-2}C_1^T C_1 - C_2^T C_2)Y_\infty + B_1B_1^T = 0 . \tag{4.69}$$

Theorem 4.6

A stabilizing controller K(s) exists such that (4.67) is satisfied if and only if

(i) $X_\infty \geq 0$

(ii) $Y_\infty \geq 0$

(iii) $\rho(X_\infty Y_\infty) < \gamma^2$

where $\rho(\cdot)$ denotes the spectral radius. When these conditions are satisfied, all stabilizing controllers are parameterized as

$$K(s) = \Phi(M_\infty(s), Q(s)) \tag{4.70}$$

where

$$M_\infty(s) = \left[\begin{array}{c|cc} \hat{A}_\infty & -Z_\infty L_\infty & Z_\infty B_2 \\ \hline F_\infty & 0 & I \\ -C_2 & I & 0 \end{array}\right] \tag{4.71}$$

$$\hat{A}_\infty \triangleq A + \gamma^{-2} B_1 B_1^T X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2 \tag{4.72}$$

$$F_\infty \triangleq -B_2^T X_\infty \tag{4.73}$$

$$L_\infty \triangleq -Y_\infty C_2^T \tag{4.74}$$

$$Z_\infty \triangleq (I - \gamma^{-2} Y_\infty X_\infty)^{-1} \tag{4.75}$$

and $Q \in \mathbf{R}H_\infty$, $\| Q \|_\infty < \gamma$.

Δ

Although the formula presented here is remarkably simple, but one should be aware that Assumption 4.4 is quite restrictive. A more general case is considerd by Glover and Doyle (1988) where some of these restrictions are relaxed at the expense of a more complicated controller, as presented below.

We still consider the same block diagram and realization for P as given above but we replace Assumption 4.4 by the following assumption.

Assumption 4.5

(i) (A, B_2, C_2) is stabilizable and detectable.

(ii) D_{12} and D_{21} have full column and row ranks respectively.

(iii) $D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$ and $D_{21} = \begin{bmatrix} 0 & I \end{bmatrix}$.

(iv) $D_{22} = 0$.

(v) $\operatorname{rank} \begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2 \quad \forall\ \omega \in \mathbf{R}$.

(vi) $\operatorname{rank} \begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2 \quad \forall\ \omega \in \mathbf{R}$.

The satisfaction of (iii) can be insured using the loop shifting technique discussed in Chapter II. Parts (v) and (vi) are usually satisfied if the realization for P is minimal.

The following matrices are defined :

$$D_{1*} = \begin{bmatrix} D_{11} & D_{12} \end{bmatrix} \tag{4.76}$$

$$D_{*1} = \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} \tag{4.77}$$

$$R = D_{1*}^T D_{1*} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix} \tag{4.78}$$

$$\tilde{R} = D_{*1} D_{*1}^T - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix} \tag{4.79}$$

$$D_{11} = \begin{bmatrix} D_{1111} & D_{1112} \\ D_{1121} & D_{1122} \end{bmatrix} \tag{4.80}$$

where $D_{1122} \in \mathbf{R}^{m_2 * p_2}$. Let X_∞ and Y_∞ be the solutions to the AREs

$$(A - B R^{-1} D_{1*}^T C_1)^T X_\infty + X_\infty (A - B R^{-1} D_{1*}^T C_1) - X_\infty B R^{-1} B^T X_\infty + C_1^T (I - D_{1*} R^{-1} D_{1*}^T) C_1 = 0 \tag{4.81}$$

$$(A - B_1 D_{*1}^T \tilde{R}^{-1} C) Y_\infty + Y_\infty (A - B_1 D_{*1}^T \tilde{R}^{-1} C)^T - Y_\infty C^T \tilde{R}^{-1} C Y_\infty + B_1 (I - D_{*1}^T \tilde{R}^{-1} D_{*1}) B_1^T = 0 \tag{4.82}$$

Using these solutions, we define

$$F \triangleq -R^{-1} \left[D_{1*}^T C_1 + B^T X_\infty \right] = \begin{array}{l} \left[\begin{array}{c} F_{11} \\ F_{12} \\ F_2 \end{array} \right] \end{array} \begin{array}{l} \uparrow (m_1 - p_2) \\ \uparrow p_2 \\ \uparrow m_2 \end{array} \tag{4.83}$$

$$H \triangleq -\left[B_1 D_{*1}^T + Y_\infty C^T \right] \tilde{R}^{-1} = \begin{array}{c} \left[\begin{array}{ccc} H_{11} & H_{12} & H_2 \end{array} \right] \\ \begin{array}{ccc} \leftrightarrow & \leftrightarrow & \leftrightarrow \\ (p_1 - m_2) & m_2 & p_2 \end{array} \end{array} \tag{4.84}$$

Theorem 4.7

For the system shown in Figure 10 with P(s) given in (4.66) satisfying Assumption 4.5 :

(a) A stabilizing controller K(s) exists such that (4.67) holds if and only if

(i) $\gamma > \max (\bar{\sigma} [D_{1111} \quad D_{1112}] , \bar{\sigma} [D_{1111}^T \quad D_{1121}^T])$ (4.85)

(ii) $X_\infty \geq 0$, $Y_\infty \geq 0$ and $\rho (X_\infty Y_\infty) < \gamma^2$. (4.86)

(b) If all conditions in (a) are satisfied, then all stabilizing controllers are given by

$$K(s) = \Phi (K_a, \Psi) \tag{4.87}$$

for arbitrary $\Psi \in \mathbf{R}H_\infty$, $\|\Psi\|_\infty < \gamma$, where

$$K(s) = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & 0 \end{array} \right] \tag{4.88}$$

in which

$$\hat{D}_{11} = -D_{1121} D_{1111}^T (\gamma^2 I - D_{1111} D_{1111}^T)^{-1} D_{1112} \tag{4.89}$$

$\hat{D}_{12} \in \mathbf{R}^{m_2 * m_2}$ and $\hat{D}_{21} \in \mathbf{R}^{p_2 * p_2}$ are any matrices satisfying

$$\hat{D}_{12}\hat{D}_{12}^T = I - D_{1121}(\gamma^2 I - D_{1111}^T D_{1111})^{-1} D_{1121}^T \tag{4.90}$$

$$\hat{D}_{21}^T \hat{D}_{21} = I - D_{1112}^T(\gamma^2 I - D_{1111} D_{1111}^T)^{-1} D_{1112} \tag{4.91}$$

$$\hat{B}_2 = (B_2 + H_{12})\,\hat{D}_{12} \tag{4.92}$$

$$\hat{C}_2 = -\hat{D}_{21}(C_2 + F_{12})\,Z \tag{4.93}$$

$$\hat{B}_1 = -H_2 + \hat{B}_2\hat{D}_{12}^{-1}\hat{D}_{11} \tag{4.94}$$

$$\hat{C}_1 = F_2 Z + \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2 \tag{4.95}$$

$$\hat{A} = A + HC + \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1 \tag{4.96}$$

$$Z = (I - \gamma^{-2} Y_\infty X_\infty)^{-1} . \tag{4.97}$$

$\triangle$

Assumption 4.5(iv) may be relaxed provided $\det(I + \hat{D}_{11} D_{22}) \neq 0$. If so, all the characterizations above still hold true except that (4.87) is replaced by

$$\tilde{K}(s) = \Phi(\tilde{K}_a, \Psi) \tag{4.98}$$

for arbitrary $\Psi \in \mathbf{R}H_\infty$, $\|\Psi\|_\infty < \gamma$, where

$$\tilde{K}_a(s) = \left[\begin{array}{c|c} \hat{A} - \hat{B}(I-M)\hat{D}^{-1}\hat{C} & \hat{B}M \\ \hline \tilde{M}\hat{C} & \hat{D}M \end{array}\right] \tag{4.99}$$

$$M = \left[I + \begin{pmatrix} D_{22} & 0 \\ 0 & 0 \end{pmatrix} \hat{D} \right]^{-1} \tag{4.100}$$

$$\tilde{M} = \left[I + \hat{D} \begin{pmatrix} D_{22} & 0 \\ 0 & 0 \end{pmatrix} \right]^{-1} . \tag{4.101}$$

B. STATE-FEEDBACK METHODS

1. Disturbance Attenuation : The H_∞ control problem with full-state feedback is considered in this section. More precisely, we are considering the block diagram shown in Figure 10 where P(s) is given by

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & 0 \\ I & 0 & 0 \end{array}\right] = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \tag{4.102}$$

$$P_{ij}(s) = C_i(\ sI-A\ )^{-1}B_j \in \mathbf{R}(s)^{p_i * m_j} \quad , \qquad i, j = 1, 2\ .$$

The problem is to determine a state-feedback controller of the form $u=Kx$ so that the disturbance w is attenuated below a prespecified level γ.

This problem has been considered by a number of authors but the earliest solution is due to Petersen (1987) and is summarized below.

Definition 4.1

For a given real constant $\gamma > 0$, the plant P in (4.102) is said to be stabilizable with disturbance attenuation γ if there exists a state feedback matrix $F \in \mathbf{R}^{m*n}$ such that

(i) the matrix $\bar{A} = A+BF$ is a stability matrix ,

(ii) $\| C_1(\ sI-\bar{A}\ )^{-1}B_1 \|_\infty \leq \gamma$.

The following condition serves as a stabilizability test (with disturbance attenuation γ) for a given system.

Condition 4.1

Suppose the positive-definite matrices $Q \in \mathbf{R}^{n*n}$ and $R \in \mathbf{R}^{m*m}$ and the constant $\gamma>0$ are given. Then the plant P(s) is said to satisfy Condition 4.1 with attenuation constant γ if there exists an $\epsilon>0$ such that the Riccati equation

$$A^TP + PA - P\ (\ \tfrac{1}{\epsilon}\ B_2R^{-1}B_2^T - \tfrac{1}{\gamma}\ B_1B_1^T\ )\ P + \tfrac{1}{\gamma}\ C_1^TC_1 + \epsilon\ Q = 0 \tag{4.103}$$

has a positive-definite solution P.

Δ

The choice of Q and R is immaterial as is proven in Petersen (1987). The theorem below provides a solution to the problem being considered.

Theorem 4.8

If P(s) satisfies Condition 4.1 with attenuation constant γ, then P(s) is stabilizable with disturbance attenuation γ. Furthermore, the required feedback gain matrix is given by

$$F = -\frac{1}{2\epsilon} R^{-1} B_2^T P \tag{4.104}$$

where P is the solution to the algebraic Riccati equation (4.103).

Δ

This procedure is remarkably simple. By applying this procedure iteratively with successively smaller values of γ, one can approach the H_∞ optimum. However, for each value of γ, a series of algebraic Riccati equations have to be solved with regard to a permissible value of ϵ. Petersen (1989) has slightly generalized and modified this method so that only a single algebraic Riccati equation needs to be solved for each value of γ.

2. Riccati Inequality : A slightly different approach to tackle the H_∞ control problem using state-feedback was introduced by Scherer (1990). Instead of the algebraic Riccati equation which is commonly used in H_∞ control problem, the Riccati inequality is used in this method. In addition to the solution to the controller, Scherer (1990) also presents a formula for the computation of the H_∞ optimum value. This method is presented below.

Again, we consider the block diagram shown in Figure 10 with P(s) given by

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C & 0 & D \\ I & 0 & 0 \end{array}\right]. \tag{4.105}$$

The following mild assumption is assumed to be satisfied.

Assumption 4.6

(i) (A, B_2) is stabilizable.

(ii) For w=0, the plant P has no zero at infinity.

The system equations corresponding to (4.105) are

$$\dot{x} = Ax + B_1 w + B_2 u \tag{4.106a}$$

$$z = Cx + Du \tag{4.106b}$$

$$y = x\,. \tag{4.106c}$$

Notice that the system described in (4.106) with dynamic compensator

$$\dot{m} = Nm + Mx \tag{4.107a}$$

$$u = Lm + Fx \tag{4.107b}$$

is equivalent to the system described by

$$\begin{bmatrix} \dot{x} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0_r \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w + \begin{bmatrix} B_2 & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{4.108a}$$

$$z = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix} + \begin{bmatrix} D & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{4.108b}$$

with static state-feedback law

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} F & L \\ M & N \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}. \tag{4.109}$$

In (4.108), r denotes the dimension of the controller state m and v is the reference input of the closed-loop system. Equations (4.108) and (4.109) may be written compactly as

$$\dot{x}_r = A_r x_r + B_{1_r} w + B_{2_r} u_r \tag{4.110a}$$

$$z \;=\; C_r x_r + D_r u_r \tag{4.110b}$$

$$u_r \;=\; F_r x_r \;. \tag{4.111}$$

The feedback matrix F_r is said to be admissible if ($A_r + B_r F_r$) is stable. We shall introduce the following notations :

$$\gamma\,(\,F_r\,) \triangleq \|\,(\,C_r + D_r F_r\,)\,(\,sI - A_r - B_{2r} F_r\,)^{-1} B_{1r}\,\|_\infty \tag{4.112}$$

$$\gamma_* \triangleq \inf\,\{\,\gamma\,(\,F_r\,) \mid r \in \mathbf{R}\,,\ F_r \text{ admissible}\,\}\;. \tag{4.113}$$

Also, we define the following transformations as admissible :

(a) static state-feedback $u = F_0 x + v$,

(b) coordinate changes in the state- or input-space ,

(c) orthogonal coordinate changes in the output space.

Proposition 4.1

There exist admissible transformations for the system (4.106) such that any extended system (4.110) has the form

$$\dot{x}_r = \begin{bmatrix} A_0 & 0 & 0 \\ * & A^- & 0 \\ * & 0 & 0_r \end{bmatrix} x_r + \begin{bmatrix} B_{10} \\ * \\ * \end{bmatrix} w + \begin{bmatrix} B_{20} & 0 & 0 \\ * & B^- & 0 \\ * & 0 & I_r \end{bmatrix} u_r \tag{4.114a}$$

$$z = \begin{bmatrix} C_0 & 0 \end{bmatrix} x_r + \begin{bmatrix} D_0 & 0 \end{bmatrix} u_r \tag{4.114b}$$

in which

$$C_0 = \begin{bmatrix} * \\ 0 \end{bmatrix} \quad \text{and} \quad D_0 = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{4.115}$$

such that A^- is stable.

$\triangle$

It is assumed that the extended system (4.110) is given without any restrictions as stated in the above proposition.

Now consider the Riccati inequality

$$R(\,X,\mu\,) \triangleq A_0 X + X A_0^T + \mu G_0 G_0^T - B_{20} B_{20}^T + X C_0^T C_0 X \leq 0 \tag{4.116}$$

where

$$\mu \triangleq \gamma^{-2} . \tag{4.117}$$

The Riccati inequality (4.116) has a solution iff there exists a solution for the corresponding Riccati equation. The unique maximal solution in which $R(X,\mu)=0$ is denoted as $X(\mu)$ and define

$$A(\mu) \triangleq A_0 + X(\mu)C_0^T C_0 \tag{4.118}$$

and

$$\mu_{max} = \sup \{ \mu \in \mathbf{R} \mid R(X,\mu)=0 \text{ has a real symmetric solution } \} \leq \infty . \tag{4.119}$$

Theorem 4.9

For any $\mu>0$, there exists an admissible static state-feedback law iff $X(\mu)$ exists and is positive definite. In that case, the admissible control law is given by

$$F_0(\mu) = -B_2^T \begin{bmatrix} X(\mu)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbf{R}^{m*n} \tag{4.120}$$

with

$$\gamma (F_0(\mu)) \leq \frac{1}{\sqrt{\mu}} . \tag{4.121}$$

$\triangle$

The above Theorem provides us with only the admissible control law. We need a way to determine the optimum value of μ, denoted μ_{max}, which corresponds to the optimal controller. The next theorem serves this purpose.

Theorem 4.10

Let $X(0)$ be computed.

(i) If $C_0(sI+A(0))^{-1}B_{10} = 0$, then $\mu_{max}=\infty$ and the optimum is never attained.

(ii) If $C_0(sI+A(0))^{-1}B_{10} \neq 0$, then

$$\mu_{max} = \| C_0(sI+A(0))^{-1}B_{10} \|_\infty^{-2} \tag{4.122}$$

and if $X(\mu_{max}) > 0$, then

$$F_0(\mu_{max}) = -B_2^T \begin{bmatrix} X(\mu_{max})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tag{4.123}$$

is the optimal controller.

Δ

In case (i) where the optimal level is not achievable, a sub-optimal controller that corresponds to $\mu_s \in (-\infty, \mu_{max}]$, $\mu_s < \infty$ can always be found. The interested reader is referred to Scherer (1990) for the details.

3. Simultaneous H_2/H_∞ Optimal Control : The problem of optimizing the H_2-norm of a transfer matrix subject to an H_∞-norm constraint of another transfer matrix has received significant attention from a number of researchers. One of its applications is the entropy maximization subject to an H_∞-norm bound, see Mustafa and Glover (1988). However, the analytical solution to the above general problem has yet been found. The problem of simultaneous H_2/H_∞ control has first been introduced and solved by Rotea and Khargonekar (1990) for the full-state feedback case. Their results are abridged below.

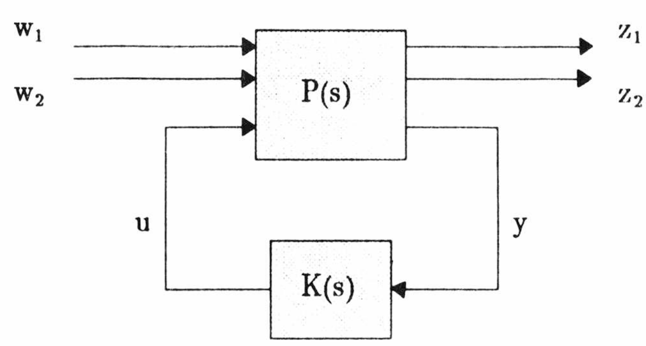

Figure 11 Block diagram for simultaneous H_2/H_∞ control

The block diagram used in this section is shown in Figure 11 where the plant P(s) is given by

$$
P(s) = \left[\begin{array}{c|ccc} A & B_1 & B_2 & B_3 \\ \hline C_1 & 0 & 0 & D_1 \\ C_2 & 0 & 0 & D_2 \\ I & 0 & 0 & 0 \end{array}\right] \tag{4.124}
$$

and the signals w, z, u and y are the same as those defined in the previous chapters. A proper, real-rational, stabilizable and detectable controller $K(s)$ which internally stabilizes $P(s)$ is said to be admissible. The closed-loop transfer functions from w_i to z_i will be denoted $T_i(s)$, where $i=1,2$.

Problem Statement

For the plant $P(s)$ defined in (4.124), find (if possible) an admissible controller $K(s)$ that achieves

$$
\inf \left\{ \| T_1(K) \|_2 : K \text{ admissible} \right\} \tag{4.125}
$$

and satisfying the normalized criterion

$$
\| T_2(K) \|_\infty < 1 \, . \tag{4.126}
$$

$\triangle$

The plant P is assumed to satisfy the following assumption.

Assumption 4.7

(i) (A, B_3) is stabilizable.

(ii) D_1 and D_2 both have full column rank.

(iii) $\begin{bmatrix} A - j\omega I & B_3 \\ C_i & D_i \end{bmatrix}$ has full column rank for every $\omega \in \mathbf{R}$ where $i=1,2$.

(iv) $D_2^T \begin{bmatrix} C_2 & D_2 \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$.

Recall that (iv) can be satisfied using the loop shifting technique discussed in Chapter II. Let X_1 be the unique solution to the algebraic Riccati equation

$$
(A - B_3 R_1 D_1^T C_1)^T X_1 + X_1 (A - B_3 R^{-1} D_1^T C_1) - X_1 B_3 R_1 B_3^T X_1 + C_1^T (I - D_1 R_1 D_1^T) C_1 = 0 \tag{4.127}
$$

where

$$R_1 = (\, D_1^T D_1 \,)^T \,. \tag{4.128}$$

Also, we make the following definitions :

$$F \;= -R_1(\, D_1^T C_1 + B_3^T X_1 \,) \tag{4.129}$$

$$\Pi_1 \;= I - B_1 B_1^{\dagger} \tag{4.130}$$

$$A_f \;= A + B_3 F \tag{4.131}$$

$$C_{2f} = C_2 + D_2 F \tag{4.132}$$

$$V_2 \;= C_2 + D_2 F \tag{4.133}$$

where $\dagger$ denotes the pseudo-inverse. If M=0, we define $M^{\dagger}=0$. In addition, we let X_2 and Y_2 be the solutions to the ARE's

$$A^T X_2 + X_2 A + X_2(\, B_2 B_2^T - B_3 B_3^T \,) X_2 + C_2^T C_2 = 0 \tag{4.134}$$

$$Y_2 A_f^T + A_f Y_2 + Y_2(\, C_{2f}^T C_{2f}) Y_2 + B_2(\, I - V_2^{\dagger} V_2 \,) B_2^T = 0 \,. \tag{4.135}$$

Theorem 4.11

For a given feedback system as depicted in Figure 11 with P as given in (4.124), there exists an admissible controller K satisfying (4.125) and (4.126) if and only if the following conditions hold :

(i) $\overline{A}_1 = A + (\, B_2 B_2^T - B_3 B_3^T \,)\, X_2$ is a stability matrix and $X_2 \geq 0$,

(ii) $\overline{A}_2 = A + (\, C_{2f}^T C_{2f} \,)\, Y_2$ is a stability matrix ,

(iii) $\rho\,(\, Y_2 X_2 \,) < 1$.

Moreover, when these conditions hold, the solution is given by

$$K(s) = \left[\begin{array}{c|c} A_0 & A_0\Sigma - \Sigma\, A_f \\ \hline H-F & F(\, I-\Sigma \,)+H\,\Sigma \end{array}\right] \tag{4.136}$$

where

$$A_0 \triangleq A + (\, I-\Sigma \,)\, B_3 H + \Sigma\, B_3 F + (\, I-\Sigma \,)\, B_2 B_2^T\, X_2 \tag{4.137}$$

$$\Sigma \triangleq Z_2 B_2 V_2^{\dagger} \Pi_1 \tag{4.138}$$

$$H \triangleq -B_3^T\, X_2 \tag{4.139}$$

$$Z_2 \triangleq (I - Y_2 X_2)^{-1} \ . \tag{4.140}$$

$\triangle$

It is also shown in Rotea and Khargonekar (1990) that there are some cases where all solutions to the simultaneous H_2/H_∞ control problem must necessarily be dynamic even though all the states are available for feedback.

4. Sensitivity For State-Feedback : The definition of sensitivity as given in equation (3.8) is meant for output feedback. In this section, we show that this definition is naturally and directly extended to the state-feedback case.

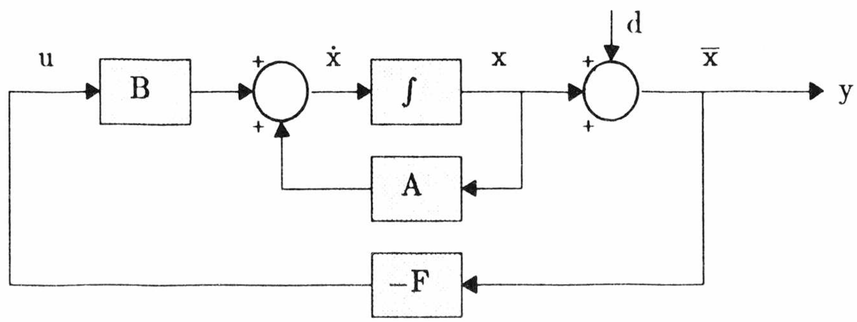

Figure 12 Block diagram for state-feedback

In Figure 12, y is the measured state vector corrupted by the measurement noise and other disturbances d. The transfer function from d to y can be derived as follows :

$$\dot{x} = Ax - BF\bar{x}$$

$$= (A - BF)x - BFd$$

$$x(s) = -\Big[(sI - A + BF)^{-1} BF \Big] d(s) \tag{4.141}$$

$$y(s) = \Big[-(sI - A + BF)^{-1} + I \Big] d(s) \ . \tag{4.142}$$

Then, the sensitivity function is naturally defined as

$$S \triangleq -(sI - A + BF)^{-1} BF + I \ . \tag{4.143}$$

It is easy to verify that this is precisely the sensitivity function defined for the output feedback case, namely

$$S \triangleq (I + GF)^{-1} . \tag{4.144}$$

C. SOME REMARKS

One can use either of the first two techniques presented in Section B to design a static H_∞ optimal or sub-optimal controller provided all the states are available for feedback. The following question immediately arises : Assuming all the states are measureable, will a dynamic H_∞ controller be better than a static H_∞ controller ? The answer is no. Khargonekar et al. (1988) proved that if γ_s is the L_∞-norm of the closed-loop transfer function from w to z with a static H_∞ controller and γ_d is the same transfer function with a dynamic H_∞ controller, then $\gamma_s = \gamma_d$. To complement this result, Scherer (1990) showed that there exists an admissible static H_∞ controller such that $\gamma_s = \gamma_*$ if and only if there exists an admissible dynamic H_∞ controller such that $\gamma_d = \gamma_*$. This concludes that dynamic H_∞ controllers offer absolutely no advantage over static H_∞ controllers.

The Hankel Approximation approach for the output feedback case has the advantage that it utilizes the balanced representation of a system which has numerous desirable numerical properties. In addition to that is the simplicity of the algorithm. However, this technique is restricted to the so-called 1-block model-matching problem in which both T_{12} and T_{21} are assumed to be square. Clearly, a large class of problems do not satisfy this assumption, for instance, the combined performance and stability requirements problem. Also, this approach does not directly provide the formula for the controller, instead it solves for the optimal Q from which the controller has to be recovered.

The H_∞ interpolation approach solves the model-matching problem in its most general form, namely the 4-block problem but the controller computation algorithm is complicated. In addition, this approach requires several preliminary treatments of the plant which raise the need for the resulting controller to be repeatedly reverse-scaled. However, the theoretical derivation of this method is intuitively clear.

The solution method based on J-lossless conjugation is remarkably simple with mild assumptions and no preliminary plant treatment. Moreover, the concepts and tools used for the derivation are clear and without the involvement of complicated theories. Also, the use of the Lyapunov solutions instead of the Riccati solutions, as common for other methods, is a distinctive feature of this technique. Although the conjugation-based method as presented in this document solves the 1-block problem only, but it should be mentioned that this method has been extended to the general 4-block case by Kimura and Kawatani (1988).

Solving the H_∞ control problem via the general distance problem approach as introduced by Doyle et al. (1989) has received a lot of attention from researchers around the world and has remained so far the most popular method used for H_∞ control. This technique solves the H_∞ control problem in its most general form with a simple algorithm. However, the concepts and mathematics used here are very involved. The many assumptions made with this algorithm require some preliminary treatments of the plant, namely the loop shifting operations, which make the reverse-transform of the controller necessary.

The disturbance attenuation method for the state-feedback H_∞ control problem is strikingly simple. However, it assumes a special form of P with no direct transmissions from w and u to z. Also, an algebraic Riccati equation has to be repeatedly solved for each value of γ.

The approach introduced by Scherer (1990) utilizes the extended plant concept which requires some preliminary operations that could be cumbersome. It also assumes that no direct transmission from $\mathbf{w}$ to z occurs. Based on the extended plant, a formula for the optimal value of γ is provided.

V. REDUCED ORDER H_∞ CONTROLLER DESIGN

A low order controller is always desirable for reasons of reduced computational burden and simplicity in hardware implementation. There are a number of different model reduction techniques available in the control literature, see Enns (1984), Haddad and Bernstein (1989), Jamshidi (1983), McFarlane et al. (1990), Moore (1981) and Prakash and Rao (1989), for example. Some of these techniques use a frequency domain approach while others concern themselves in the time domain.

This chapter is divided into four sections. In Section A, we briefly introduce three balanced truncation model reduction methods and discuss the modified procedure in the case where we have an unstable plant. A new balanced truncation model reduction method with reduced error bound is proposed in Section B. The general formula for the model reduction error as well as its formulae at zero and infinite frequencies are explicitly derived. Also furnished are the necessary and sufficient condition for stability of the reduced order model and a set of guidelines for choosing the new parameter in order to achieve a lower error bound. The advantages of this proposed technique over those existing techniques are highlighted. Section C is devoted to the discussion of the order reduction alternatives, namely plant or controller model reduction. In Section D, the concept of combined state and output feedback H_∞ controller design is introduced. The different closed-loop structure incurred is analyzed and the formulae for sensitivity and closed-loop transfer function are derived.

A. REVIEW OF TRUNCATION METHODS

1. Balancing Model Reduction : This method was first proposed by Moore (1981) and has been extended and modified by several authors, Enns (1984a,b) and Prakash and Rao (1989). The method relies on a balanced state-space realization of a given system.

Let a balanced realization of a system G(s) be given by

$$G(s) = (A, B, C, D) \tag{5.1}$$

with controllability/observability gramian equal to

$$\Sigma = \operatorname{diag} \{ \sigma_i \} = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \tag{5.2}$$

in which

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \qquad \text{and} \qquad \sigma_r \gg \sigma_{r+1} \tag{5.3}$$

where $\Sigma_1 \in \mathbf{R}^{r*r}$, $\Sigma_2 \in \mathbf{R}^{(n-r)*(n-r)}$, r<n and n is the order of the system. Let A, B and C be partitioned compatibly with Σ as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}. \tag{5.4}$$

Then, a reduced order model for G can be obtained as

$$G_r(s) = (A_{11}, B_1, C_1, D) . \tag{5.5}$$

The upper bound for the model reduction error introduced by this truncation is proved by Enns (1984a,b) to be

$$E_\infty \triangleq \| G(s) - G_r(s) \|_\infty \leq 2 \operatorname{tr} [\Sigma_2] = 2 \sum_{i=r+1}^{n} \sigma_i . \tag{5.6}$$

The motivation for the use of the L_∞-norm error criterion (as opposed to the classical integral square of some function) is that it is the magnitude of the highest peak of the error function over frequency that is important due to stability requirement (see Ridgely (1986)) .

<u>**2. Low Frequency Approximation :**</u> It is discovered by Prakash and Rao (1989) that the method proposed in Section 1 above introduces a model

reduction error which is large in low frequencies and small in high frequencies. This is not desirable for control applications since most practical systems do operate in low frequency ranges. Prakash and Rao (1989) proposed a reduction technique which offers a good match at low frequencies while allowing the reduction error to be higher in high frequencies. This method is summarized below.

Let the balanced realization of a system and its controllability/ observability gramian be given as in (5.1), (5.2) and (5.3). Also let A, B and C be partitioned as in (5.4). Define

$$A_r \triangleq A_{11} - A_{12}\, A_{22}^{-1}\, A_{21} \tag{5.7}$$

$$B_r \triangleq B_1 - A_{12}\, A_{22}^{-1}\, B_2 \tag{5.8}$$

$$C_r \triangleq C_1 - C_2\, A_{22}^{-1}\, A_{21} \tag{5.9}$$

$$D_r \triangleq D - C_2\, A_{22}^{-1}\, B_2\ . \tag{5.10}$$

Then, a reduced-order model for G is obtained as

$$G_r(s) = (\ A_r, B_r, C_r, D_r\ )\ . \tag{5.11}$$

It has been proved that the reduction error equals to zero at zero frequency and approach a constant finite value as the frequency tends to infinity. Also, this reduced order model is guaranteed to be stable and minimal. The model reduction error bound is the same as the one given in (5.6).

3. Frequency Weighted Balancing : A frequency weighted generalization of the balancing model reduction technique discussed in Section 1 has been introduced by Enns (1984a,b). The idea is to employ two frequency dependent weighting matrices, one at the input and the other at the output of the system, before obtaining the balanced realization. Once the balanced realization is obtained, the state truncation method in Section 1 is applied directly. This approach has the advantage that, by properly choosing the weighting matrices, the reduction error can be made smaller in a particular frequency range while it is allowed to be larger in other frequency ranges.

Let a strictly proper system G with n states be given as

$$G(s) = (A, B, C, 0) . \tag{5.12}$$

Two input and output weighting matrices, $W_i(s)$ and $W_o(s)$ having n_i and n_o states respectively with the realizations

$$W_i(s) = (A_i, B_i, C_i, D_i) \quad \text{and} \quad W_o(s) = (A_o, B_o, C_o, D_o) \tag{5.13}$$

are chosen. The cascade realizations of GW_i and W_oG are then formed :

$$GW_i = (A_c, B_c, C_c, D_c) \tag{5.14}$$

$$W_oG = (A_o, B_o, C_o, D_o) . \tag{5.15}$$

The controllability gramian L_c of the input-weighted system GW_i and the observability gramian L_o of the output-weighted system W_oG are obtained by solving their defining Lyapunov equations, respectively. Then the frequency weighted controllability and observability gramians $\hat{L}_c$ and $\hat{L}_o$ are obtained by extracting the upper left nxn sub-matrix of L_c and L_o correspondingly :

$$L_c = \begin{bmatrix} \hat{L}_c & * \\ * & * \end{bmatrix} , \qquad L_o = \begin{bmatrix} \hat{L}_o & * \\ * & * \end{bmatrix} . \tag{5.16}$$

The required similarity transformation matrix T is calculated using the eigenvalue-eigenvector decomposition of L_cL_o as follows :

$$L_cL_o = T\Lambda T^{-1} . \tag{5.17}$$

Then, the frequency weighted balanced realization for G is obtained as

$$G_{wb}(s) = (\hat{A}, \hat{B}, \hat{C}, 0) = (T^{-1}AT, T^{-1}B, CT, 0) . \tag{5.18}$$

The model reduction technique discussed in Section 1 can be applied directly to the realization in (5.18).

The model reduction error for this case is defined as

$$E_\infty \triangleq \| W_o(s) [G(s) - G_r(s)] W_i(s) \|_\infty . \tag{5.19}$$

For the case where either $W_o = I$ or $W_i = I$, the reduced order model is guaranteed to be stable but for the general case of non-unity weightings, no guarantee on the stability can be made.

4. <u>**Unstable Systems :**</u> Reducing the order of the plant is often necessary even if the open loop plant contains unstable poles and for the purpose of controller design, the number of unstable poles in the full order model must be retained in the reduced order model. Since all three model reduction techniques discussed above do require the open loop plant be asymptotically stable, these methods can not be applied directly to unstable plants. However, we may circumvent this difficulty by decomposing the plant into two portions, a stable portion and an anti-stable portion, and then reduce the order of the stable portion only, leaving the anti-stable portion untouched. This procedure was recommended by Enns (1984b).

Given an unstable full order model $G = (A, B, C, D)$, determine similarity transformation matrix T such that

$$\hat{G} = (TAT^{-1}, TB, CT^{-1}, D) = \left(\begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \begin{bmatrix} D \end{bmatrix} \right) \tag{5.20}$$

in which A_s is stable and A_u is anti-stable. We then obtain a reduced order model for (A_s, B_1, C_1, D) using any technique described above and the reduced order model for G is given by

$$G_r(s) = \left(\begin{bmatrix} A_{sr} & 0 \\ 0 & A_u \end{bmatrix}, \begin{bmatrix} B_{1r} \\ B_2 \end{bmatrix}, \begin{bmatrix} C_{1r} & C_2 \end{bmatrix}, \begin{bmatrix} D \end{bmatrix} \right) . \tag{5.21}$$

The resulting modeling error will be that due to the order reduction of the stable part only, which is

$$E_\infty \triangleq \| G(s) - G_r(s) \|_\infty$$

$$= \| G_s(s) + G_u(s) - G_{sr}(s) - G_u(s) \|_\infty$$

$$= \| G_s(s) - G_{sr}(s) \|_\infty$$

$$\leq 2 \sum_{i=r+1}^{m} \sigma_i \tag{5.22}$$

where m is the dimension of A_s and r is the dimension of A_{sr}.

B. BALANCED TRUNCATION WITH REDUCED ERROR BOUND

After reviewing some balanced truncation methods in Section A, we now propose a new technique for obtaining a reduced order model which embraces both method 1 and method 2 in the above section. In fact, the balancing model reduction method introduced by Moore (1981) and the low frequency approximation method proposed by Prakash and Rao (1989) fall out to be two special cases of the below-proposed method. The following technique may be viewed as a generalization of the above two methods.

Let the n^{th} order internally balanced system G be partitioned as

$$G(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \tag{5.23}$$

where $A_{11} \in \mathbf{R}^{r*r}$. Accordingly, the controllability/observability gramian of G is partitioned conformally with (5.23) as

$$\Sigma = \mathrm{diag}(\sigma_i) = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \tag{5.24}$$

where $\Sigma_1 \in \mathbf{R}^{r*r}$, r<n and $\sigma_{r+1} \ll \sigma_r$. The system equations that correspond to the

system described in (5.23) are

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \tag{5.25}$$

$$\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) \tag{5.26}$$

$$y(t) = C_1x_1(t) + C_2x_2(t) + Du(t)\ . \tag{5.27}$$

We approximate the states of the weaker sub-system as

$$X_2(s) \approx -\delta\, A_{22}^{-1}A_{21}X_1(s) - \delta\, A_{22}^{-1}B_2U(s) \tag{5.28}$$

where δ is a positive real constant and $X_1(s)$, $X_2(s)$ and $U(s)$ are the Laplace transforms of the corresponding state and input vectors. By Laplace transforming equations (5.25) and (5.27) and making the substitution (5.28), we obtain the following reduced order model :

$$G_r(s) = (\ A_r, B_r, C_r, D_r\ ) \tag{5.29}$$

where

$$A_r = A_{11} - \delta\, A_{12}A_{22}^{-1}A_{21} \tag{5.30}$$

$$B_r = B_1 - \delta\, A_{12}A_{22}^{-1}B_2 \tag{5.31}$$

$$C_r = C_1 - \delta\, C_2A_{22}^{-1}A_{21} \tag{5.32}$$

$$D_r = D - \delta\, C_2A_{22}^{-1}B_2\ . \tag{5.33}$$

We shall now explore some properties of this new reduced order model. The following theorem provides a condition for the choice of δ for which the proposed reduced order model is stable.

Theorem 5.1

Let $G_r(s)$ be the reduced order model as given in (5.29)–(5.33) and define

$$\overline{A} \triangleq A_{12}\Sigma_2A_{22}^{-T}A_{12}^{T} \tag{5.34}$$

$$Q \triangleq B_rB_r^{T} + \delta\,(\delta-1)\,(\,\overline{A}+\overline{A}^{T}\,) \tag{5.35}$$

where δ is a positive real constant. Then $G_r(s)$ is guaranteed to be stable if and only if Q is positive definite.

Proof

Using the partitions in (5.23) and (5.24), the Lyapunov equation that corresponds to the original balanced system can be dissociated into the following four Lyapunov equations :

$$A_{11}\Sigma_1 + \Sigma_1 A_{11}^T = -B_1 B_1^T \tag{5.36}$$

$$A_{12}\Sigma_2 + \Sigma_1 A_{21}^T = -B_1 B_2^T \tag{5.37}$$

$$A_{21}\Sigma_1 + \Sigma_2 A_{12}^T = -B_2 B_1^T \tag{5.38}$$

$$A_{22}\Sigma_2 + \Sigma_2 A_{22}^T = -B_2 B_2^T \ . \tag{5.39}$$

Using expression (5.31) and the substitution of the above four Lyapunov equations, we have

$$-B_r B_r^T = -(\, B_1 - \delta\, A_{12}A_{22}^{-1}B_2 \,)\,(\, B_1 - \delta\, A_{12}A_{22}^{-1}B_2 \,)^T$$

$$= (\, A_{11} - A_{12}\delta\, A_{22}^{-1}A_{21} \,)\Sigma_1 + \Sigma_1(\, A_{11} - A_{12}\delta\, A_{22}^{-1}A_{21} \,)^T$$

$$+\ \delta(\delta-1)\Big[\, A_{12}\Sigma_2 A_{22}^{-1}A_{12}^T + A_{12}A_{22}^{-1}\Sigma_2 A_{12}^T \,\Big] \tag{5.40}$$

$$= A_r\Sigma_1 + \Sigma_1 A_r^T + \delta(\delta-1)\big[\, \bar{A} + \bar{A}^T \,\big] \, . \tag{5.41}$$

Rearranging (5.41) into the Lyapunov equation form yields

$$A_r\Sigma_1 + \Sigma_1 A_r^T = -\Big[\, B_r B_r^T + \delta(\delta-1)\,(\, \bar{A} + \bar{A}^T \,)\,\Big] \, . \tag{5.42}$$

Recall that Σ_1 is diagonal with the first r singular values of the full-order model as its diagonal elements and thus is positive definite. Since the right-hand-side of equation (5.42) is symmetric, by the Lyapunov theorem, the assertion of the above theorem follows.

□

Of course, in the development of Theorem 5.1, had we started with the second form of the Lyapunov equation, namely

$$A^T\Sigma + \Sigma A = -C^T C \ , \tag{5.43}$$

then we would have gotten a different but equivalent result. This dual version of

Theorem 5.1 is stated as a collorary below.

Collorary 5.1

Let $G_r(s)$ be the reduced order model as given in (5.29)–(5.33) and define

$$\overline{\overline{A}} \triangleq A_{21}^T A_{22}^{-T} \Sigma_2 A_{21} \tag{5.44}$$

$$\overline{Q} \triangleq C_r^T C_r + \delta(\delta-1) \left(\overline{\overline{A}}^T + \overline{\overline{A}} \right) \tag{5.45}$$

where δ is a positive real constant. Then $G_r(s)$ is guaranteed to be stable if and only if $\overline{Q}$ is positive definite.

Proof

The development is exactly parallel with the proof of Theorem 5.1 and thus is omitted.

□

We now set out to derive the upper bound for the model reduction error. Writing equation (5.28) and the Laplace transform of (5.25) in matrix notation, we get

$$\begin{bmatrix} sI-A_{11} & -A_{12} \\ -A_{21} & -\frac{1}{\delta}A_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U \,. \tag{5.46}$$

Laplace transforming equation (5.27) and using (5.46) and the formula for the inverse of block matrices given in the Appendix, we arrive at

$$G_r(s) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} sI-A_{11} & -A_{12} \\ -A_{21} & -\frac{1}{\delta}A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D \tag{5.47}$$

$$= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} \alpha+\alpha A_{12}\theta^{-1}A_{21}\alpha & \alpha A_{12}\theta^{-1} \\ \theta^{-1}A_{21}\alpha & \theta^{-1} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D \tag{5.48}$$

where the definitions

$$\alpha \triangleq (sI-A_{11})^{-1} \tag{5.49}$$

$$\theta \triangleq -\tfrac{1}{\delta} A_{22} - A_{21}\alpha A_{12} \tag{5.50}$$

have been used. The same formula for inverse of block matrices is applied to the original system yielding

$$G(s) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} sI-A_{11} & -A_{12} \\ -A_{21} & sI-A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D \tag{5.51}$$

$$= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} \alpha+\alpha\, A_{12}\beta^{-1}A_{21}\alpha & \alpha\, A_{12}\beta^{-1} \\ \beta^{-1}A_{21}\alpha & \beta^{-1} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D \tag{5.52}$$

in which β is defined as

$$\beta \triangleq sI - A_{22} - A_{21}\alpha\, A_{12} \,. \tag{5.53}$$

The reduction error is defined to be the difference between the reduced model and the original model which is given by

$$\Delta(s) \;= G(s) - G_r(s) \tag{5.54}$$

$$= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} \alpha\, A_{12}(\beta^{-1}-\theta^{-1})\, A_{21}\alpha & \alpha\, A_{12}(\beta^{-1}-\theta^{-1}) \\ (\beta^{-1}-\theta^{-1})\, A_{21}\alpha & (\beta^{-1}-\theta^{-1}) \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{5.55}$$

$$= C_1\alpha\, A_{12}(\beta^{-1}-\theta^{-1})\, A_{21}\alpha B_1 + C_2(\beta^{-1}-\theta^{-1})\, A_{21}\alpha B_1 + C_1\alpha\, A_{12}(\beta^{-1}-\theta^{-1})B_2 + C_2(\beta^{-1}-\theta^{-1})B_2 \tag{5.56}$$

$$= C_1\alpha\, A_{12}(\beta^{-1}-\theta^{-1})\,(A_{21}\alpha B_1+B_2) + C_2(\beta^{-1}-\theta^{-1})\,(A_{21}\alpha B_1+B_2) \tag{5.57}$$

$$= (\, C_1\alpha\, A_{12}+C_2)\,(\beta^{-1}-\theta^{-1})\,(\, A_{21}\alpha B_1+B_2\,) \,. \tag{5.58}$$

If we define

$$\overline{C} \triangleq C_1\alpha\, A_{12} + C_2 \tag{5.59}$$

$$\Omega \triangleq \beta^{-1} - \theta^{-1} \tag{5.60}$$

$$\overline{B} \triangleq A_{21}\alpha B_1 + B_2 \,, \tag{5.61}$$

then equation (5.58) can be written in a compact form as

$$\Delta(s) \;= \overline{C}\Omega\overline{B} \,. \tag{5.62}$$

The error bound, which is the maximum singular value of Δ, is thus given by

$$\overline{\sigma}[\,\Delta\,] = \left[\,\lambda_{max}\left\{\,(\,\overline{C}\Omega\overline{B}\,)\,(\,\overline{B}^H\,\Omega^H\,\overline{C}^H\,)\,\right\}\,\right]^{\frac{1}{2}} \tag{5.63}$$

where the superscript H stands for 'Hermitian' (complex conjugate transpose). We first consider the reduction-by-one case, i.e. r=n−1. In this case, A_{22} is a scalar, and so are Σ_2 ($=\sigma_n$), θ, β, Ω, $\overline{C}$ and $\overline{B}$. Hence, equation (5.63) is reduced to

$$\overline{\sigma}[\,\Delta\,] = \left[\,(\,\Omega\Omega^*\,)\,(\,\overline{B}\,\overline{B}^*\,)\,(\,\overline{C}\,\overline{C}^*\,)\,\right]^{\frac{1}{2}} \tag{5.64}$$

where '*' denotes the complex conjugate operation. From (5.61), we have

$$\begin{aligned}\overline{B}\,\overline{B}^H &= (\,A_{21}\alpha B_1 + B_2\,)\,(\,B_1^T\alpha^H A_{21}^T + B_2^T\,) \\ &= A_{21}\alpha B_1 B_1^T \alpha^H A_{21}^T + A_{21}\alpha B_1 B_2^T + B_2 B_1^T \alpha^H A_{21}^T + B_2 B_2^T\,.\end{aligned} \tag{5.65}$$

Substituting (5.36)−(5.39) into (5.65) and simplifying, we arrive at

$$\overline{B}\,\overline{B}^H = -A_{21}\,[\,\chi^H + \chi\,]\,A_{21}^T - \Sigma_2\,[\,\Psi + \Psi^H\,] \tag{5.66}$$

where

$$\chi \triangleq (\,I + \alpha\,A_{11}\,)\Sigma_1\alpha^H \tag{5.67}$$

$$\Psi \triangleq A_{21}\alpha\,A_{12} + A_{22}\,. \tag{5.68}$$

From (5.49), we have

$$\alpha^{-1} = sI - A_{11} \tag{5.69}$$

which gives

$$I + \alpha\,A_{11} = sI\alpha\,. \tag{5.70}$$

Also, from (5.53), we have

$$A_{21}\alpha\,A_{12} + A_{22} = sI - \beta\,. \tag{5.71}$$

Using (5.70) and (5.71) with s=jω, we obtain (remember that Σ_1 is diagonal and real and β is a scalar)

$$\chi = j\omega I\alpha\Sigma_1\alpha^H \tag{5.72}$$

$$\chi H = -j\omega I\alpha\Sigma_1\alpha^H \tag{5.73}$$

$$\Psi = j\omega I - \beta \tag{5.74}$$

$$\Psi^{\mathrm{H}} = -\mathrm{j}\omega \mathrm{I} - \beta^* \ . \tag{5.75}$$

Hence, equation (5.66) boils down to

$$\overline{\mathrm{B}}\,\overline{\mathrm{B}}^{\mathrm{H}} = \Sigma_2(\ \beta+\beta^*\ ) \tag{5.76}$$

$$= 2\ \Sigma_2 \Re(\beta) \tag{5.77}$$

$$= \overline{\mathrm{B}}\,\overline{\mathrm{B}}^{*} \tag{5.78}$$

where $\Re(\cdot)$ denotes the real part of the complex number. A similar development leads to the results

$$\overline{\mathrm{C}}^{\mathrm{H}}\,\overline{\mathrm{C}} = 2\ \Sigma_2 \Re(\beta) \tag{5.79}$$

$$= \overline{\mathrm{C}}^{*}\,\overline{\mathrm{C}}\ . \tag{5.80}$$

With these results, equation (5.64) reduces to

$$\overline{\sigma}[\ \Delta\ ] = \Big[\ 4\ |\Omega|^2 \sigma_n^2\ \{\ \Re(\beta)\ \}^2\ \Big]^{\frac{1}{2}} \tag{5.81}$$

$$= 2\ \sigma_n\ \big|\Omega\ \Re(\beta)\big|\ . \tag{5.82}$$

We now proceed by defining

$$\mathrm{A}_{21}\alpha\ \mathrm{A}_{12} \triangleq \mathrm{a}+\mathrm{jb}\ . \tag{5.83}$$

Consider the definition of θ in (5.50). Using the definition (5.83), we may explicitly write out the real and imaginary parts of θ as

$$\theta = (-\tfrac{1}{\delta}\mathrm{A}_{22}-\mathrm{a}) - \mathrm{jb} \tag{5.84}$$

$$= \mathrm{c}-\mathrm{jb} \tag{5.85}$$

where

$$\mathrm{c} \triangleq -(\ \tfrac{1}{\delta}\mathrm{A}_{22}+\mathrm{a}\ )\ . \tag{5.86}$$

Similar treatment is applied to β in (5.53) like so :

$$\beta = -\mathrm{A}_{22}-\mathrm{a}+\mathrm{j}(\omega-\mathrm{b}) \tag{5.87}$$

$$= \mathrm{d} + \mathrm{j}(\omega-\mathrm{b}) \tag{5.88}$$

where

$$\mathrm{d} \triangleq -(\ \mathrm{A}_{22}+\mathrm{a}\ )\ . \tag{5.89}$$

Now, we use the definition of Ω in (5.60) to write

$$\Omega = \frac{1}{d+j(\omega-b)} - \frac{1}{c-jb}$$

$$= \frac{c-d-j\omega}{[\,d+j(\omega-b)\,]\,(c-jb)}$$

$$= \frac{(c-d)-j\omega}{[\,dc+b(\omega-b)\,]+j\,[\,c(\omega-b)-db\,]}\,. \tag{5.90}$$

Notice that $\Re(\beta)=d$, we have

$$|\Omega\, d| = \left[\frac{(c-d)^2d^2+\omega^2d^2}{[\,dc+b(\omega-b)\,]^2+[\,c(\omega-b)-db\,]^2}\right]^{\frac{1}{2}}\,. \tag{5.91}$$

After expanding the denominator, collecting and factoring terms, we arrived at

$$|\Omega\, d| = \left[\frac{(c-d)^2d^2+\omega^2d^2}{(b^2+c^2)\,[\,d^2+(\omega-b)^2\,]}\right]^{\frac{1}{2}}\,. \tag{5.92}$$

Replacing $|\Omega\Re(\beta)|$ in (5.82) by the expression in (5.92), we obtain the general expression for the upper bound of the model reduction error :

$$\overline{\sigma}[\,\Delta\,] = 2\,\sigma_n \left[\frac{(c-d)^2d^2+\omega^2d^2}{(b^2+c^2)\,[\,d^2+(\omega-b)^2\,]}\right]^{\frac{1}{2}}\,. \tag{5.93}$$

Although the formal proof is not available at this time, but we claim that

$$\overline{\sigma}[\,\Delta\,] \le 2\,\delta\sigma_n \qquad \forall\omega\,. \tag{5.94}$$

This claim is supported by many numerical examples.

We shall now investigate the behavior of the error bound as ω tends to infinity. The result is stated in the following lemma.

Lemma 5.1

As ω tends to infinity, the upper bound of the model reduction error as given in (5.82) is geven by

$$\lim_{\omega \to \infty} \bar{\sigma}[\,\Delta\,] = 2\,\delta\sigma_n\ . \tag{5.95}$$

Proof

From the definition of β in (5.53) and θ in (5.50), we have

$$\beta(\infty) = -A_{22} + j\infty \quad , \quad \beta(\infty)^{-1} = 0 - j\,0\ , \tag{5.96}$$

$$\theta(\infty) = -\tfrac{1}{\delta}\,A_{22} \quad , \quad \theta(\infty)^{-1} = -\delta\,A_{22}^{-1}\ . \tag{5.97}$$

Substituting these into the definition of Ω in (5.60) yields

$$\Omega(\infty) = \delta\,A_{22}^{-1}\ . \tag{5.98}$$

Using the expression given in (5.82), we have

$$\lim_{\omega \to \infty} \bar{\sigma}[\,\Delta(j\omega)\,] = \lim_{\omega \to \infty} 2\,\sigma_n \left|\delta\,A_{22}^{-1}(-A_{22})\right| \tag{5.99}$$

$$= 2\,\delta\sigma_n\ . \tag{5.100}$$

□

The investigation of the upper bound of the model reduction error at zero frequency shows that it is a non-zero finite value as given in the next lemma.

Lemma 5.2

The upper bound of the model reduction error at zero frequency is given by

$$\bar{\sigma}[\,\Delta\,] = 2\,\sigma_n \left| \frac{(1-\frac{1}{\delta})\,A_{22}}{-\frac{1}{\delta}A_{22} + A_{21}A_{11}^{-1}A_{12}} \right|\ . \tag{5.101}$$

Proof

From their corresponding definition, we obtain

$$\beta(0) = -A_{22} + A_{21}A_{11}^{-1}A_{12} \tag{5.102}$$

$$\theta(0) = -\tfrac{1}{\delta}A_{22} + A_{21}A_{11}^{-1}A_{12}\ . \tag{5.103}$$

Recall that we are still working with the case r=n−1 which means that A_{22} is a scalar. Hence

$$\beta^{-1}(0) = \frac{1}{\beta(0)} \qquad \text{and} \qquad \theta^{-1}(0) = \frac{1}{\theta(0)} \tag{5.104}$$

and $\Omega(0)$ is obtained, after some algebra, as

$$\Omega(0) = \frac{(1-\frac{1}{\delta})\,A_{22}}{(-A_{22}+A_{21}A_{11}^{-1}A_{12})\,(-\frac{1}{\delta}A_{22}+A_{21}A_{11}^{-1}A_{12})} \quad . \tag{5.105}$$

Substituting (5.105) and (5.102) into (5.82) yields the expression (5.101).

□

We now have arrived at the central result of this section. We show that, by appropriately choosing δ, the maximum model reduction error can be reduced to a level lower than that obtained by either Moore's or Prakash and Rao's methods. In particular, we show that the following inequality

$$\overline{\sigma}[\,\Delta\,] < \overline{\sigma}[\,\Delta\,]_{c=d} \leq 2\,\sigma_n \tag{5.106}$$

can be made to be true by correctly selecting the value of δ.

Lemma 5.3

For the special case when c=d, the model reduction error as given in (5.93) is bounded by

$$\overline{\sigma}[\,\Delta\,]_{c=d} \leq 2\,\sigma_n \;. \tag{5.107}$$

Proof

From equation (5.93), for c=d, we have

$$\overline{\sigma}[\,\Delta\,]_{c=d} = 2\,\sigma_n \left[\frac{\omega^2 d^2}{(\,b^2+d^2\,)\,[\,d^2+(\omega-b)^2\,]}\right]^{\frac{1}{2}} \tag{5.108}$$

$$= 2\,\sigma_n \left[\frac{\omega^2 d^2}{(\,d^2+b^2-b\omega\,)^2+\omega^2 d^2}\right]^{\frac{1}{2}} \tag{5.109}$$

$$\leq 2\,\sigma_n \left[\frac{\omega^2 d^2}{\omega^2 d^2}\right]^{\frac{1}{2}} = 2\,\sigma_n \;. \tag{5.110}$$

□

Theorem 5.2 (Necessity)

For the inequality

$$\overline{\sigma}[\,\Delta\,] < \overline{\sigma}[\,\Delta\,]_{c=d} \tag{5.111}$$

to hold, the following two necessary conditions must be satisfied :

(i) c>d

(ii) $0<\delta<1$.

Proof

(i) We first assume that the inequality (5.111) is true. With this assumption, we have

$$2\,\sigma_n \left[\frac{(c-d)^2 d^2+\omega^2 d^2}{(b^2+c^2)\,[d^2+(\omega-b)^2]}\right]^{\frac{1}{2}} < 2\,\sigma_n \left[\frac{\omega^2 d^2}{(b^2+d^2)\,[d^2+(\omega-b)^2]}\right]^{\frac{1}{2}} \tag{5.112}$$

$$\frac{(c-d)^2+\omega^2}{b^2+c^2} < \frac{\omega^2}{b^2+d^2} \tag{5.113}$$

$$\Rightarrow \quad \frac{\omega^2}{b^2+c^2} < \frac{\omega^2}{b^2+d^2} \tag{5.114}$$

$$\Rightarrow \quad c > d\,. \tag{5.115}$$

(ii) This condition follows directly from Lemmas 5.1 and 5.3.

□

Note that Theorem 5.2 gives only the necessary conditions. The sufficiency of these conditions is currently under investigation.

We shall now investigate under what circumstances that both the necessary conditions stated in Theorem 5.2 could be satisfied. From the definitions in (5.86) and (5.89), we have

$$c-d = (\,1-\tfrac{1}{\delta}\,)\,A_{22}\,. \tag{5.116}$$

The case c>d implies that the right hand side of equation (5.116) is positive.

There are four cases for this to be true which are tabulated in Table I.

From Table I, we conclude that the model reduction error can be

Table I Choice of δ for reduced error bound

A_{22}	δ	c−d	result
positive	$\delta>1$	positive	c>d
positive	$0<\delta<1$	negative	c<d
negative	$\delta>1$	negative	c<d
negative	$0<\delta<1$	positive	c>d

reduced if A_{22} is negative. For the case $A_{22}>0$, the lowest achievable model reduction error is $2\sigma_n$ which is attained when $\delta=1$. Although no proof is available, we note here that a large number of numerical examples show that the magnitude of the decrement in model reduction error in one frequency range is equal to the magnitude of the increment in model reduction error in the other frequency range, or more precisely,

$$\lim_{\omega\to\infty} \overline{\sigma}[\,\Delta(j\omega)\,] + \overline{\sigma}[\,\Delta(0)\,] = \overline{\sigma}[\,\Delta(0)\,]_{\delta=0} \tag{5.117}$$

$$= \lim_{\omega\to\infty} \overline{\sigma}[\,\Delta(j\omega)\,]_{\delta=1} \tag{5.118}$$

$$= 2\,\sigma_n\,. \tag{5.119}$$

As is well-known, every physical system has at least one zero at infinity which implies that the transfer function of any physical system will asymptotically approach some finite values as the radiant frequency ω tends to infinity. With this in mind and by looking at the definition of the model reduction error as given in (5.54), we would expect $\Delta(j\omega)$ to approach $D-D_r$ as ω approaches infinity. This, in fact, is the case and is proven below.

Lemma 5.4

As $\omega\to\infty$, the model reduction error is given by

$$\lim_{\omega \to \infty} \bar{\sigma}[\,\Delta(j\omega)\,] = \bar{\sigma}[\,D - D_r\,] \tag{5.120}$$

where $\Delta(j\omega)$ and D_r are defined in (5.54) and (5.33) respectively.

Proof

From their corresponding definitions in (5.49), (5.59), (5.61), (5.53), (5.50), (5.60) and (5.62), we have

$$\alpha(\infty) = 0 \quad , \tag{5.121}$$

$$\overline{C}(\infty) = C_2 \quad , \quad \overline{B}(\infty) = B_2 \quad , \tag{5.122}$$

$$\beta(\infty) = \infty \quad , \quad \beta(\infty)^{-1} = 0 \quad , \tag{5.123}$$

$$\theta(\infty) = -\tfrac{1}{\delta}\, A_{22} \quad , \quad \theta(\infty)^{-1} = -\delta\, A_{22}^{-1} \quad , \tag{5.124}$$

$$\Omega(\infty) = \delta\, A_{22}^{-1} \tag{5.125}$$

and

$$\Delta(\infty) = \overline{C}(\infty)\, \Omega(\infty)\, \overline{B}(\infty) \tag{5.126}$$

$$= C_2 \delta\, A_{22}^{-1} B_2 \tag{5.127}$$

$$= D - D_r\ . \tag{5.128}$$

Since singular values of any given system are unique, the following relationship immediately follows :

$$\lim_{\omega \to \infty} \bar{\sigma}[\,\Delta(j\omega)\,] = \bar{\sigma}[\,D - D_r\,]\ . \tag{5.129}$$

□

Notice that the left hand side of equations (5.95) and (5.129) are identical and thus, equation (5.129) provides an alternative formula for the calculation of the spectral norm of the model reduction error.

The results presented above are for the reduction-by-one case. Unfortunately, for the general case where $r < n-1$, no useful and informative expression for the error bound could be derived. This is due to the fact that the reduced order model obtained using this proposed technique *is not* internally balanced, except for the cases $\delta = 1$ and $\delta = 0$ as is clear from equation (5.42).

Hence, the error bound has to be derived for each case when $r<n-1$.

Finally, we shall establish the uniqueness property of this proposed reduced order model.

Theorem 5.3 (Uniqueness)

The reduced order model obtained using this proposed technique is unique in terms of input-output behavior.

Proof

Glover (1984) showed that the similarity transformation connecting two balanced realizations is characterized by

$$T\Sigma = \Sigma T \tag{5.130}$$

where Σ is the controllability/observability gramian. Prakash and Rao (1989) have established that if Σ and T are partitioned mutually conformal as

$$\Sigma = \mathrm{diag}(\ \Sigma_1,\Sigma_2\ ) \tag{5.131}$$

$$T = \begin{bmatrix} T_1 & T_3 \\ T_4 & T_2 \end{bmatrix}, \tag{5.132}$$

then $T_3 = T_4 = 0$. Hence, applying a similarity transformation of the form

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \tag{5.133}$$

to the realization (5.23), we obtain

$$TGT^{-1} = \left[\begin{array}{cc|c} T_1A_{11}T_1^{-1} & T_1A_{12}T_2^{-1} & T_1B_1 \\ T_2A_{21}T_1^{-1} & T_2A_{22}T_2^{-1} & T_2B_2 \\ \hline C_1T_1^{-1} & C_2T_2^{-1} & D \end{array}\right]. \tag{5.134}$$

Employing the proposed model reduction technique to the realization (5.134), it is easy to verify that the reduced order model is given by

$$\overline{G}_r(s) = (\ \overline{A}_r, \overline{B}_r, \overline{C}_r, \overline{D}_r\ ) \tag{5.135}$$

$$= (\, T_1 A_r T_1^{-1},\, T_1 B_r,\, C_r T_1^{-1},\, D_r \,) \tag{5.136}$$

$$= T_1 G_r T_1^{-1} \tag{5.137}$$

which preserves the input-output behavior of the system.

□

We remark here that for the two special cases where δ=0 and δ=1, this proposed reduced order model reduces to Moore's model and Prakash and Rao's model, respectively. By introducing a parameter δ, we allow ourselves an extra degree of freedom in which the spectral norm of the model reduction error can be adjusted according to our needs. In other words, by varying δ, the relative magnitude of the model reduction error in low and high frequency ranges can be altered. In particular, if we choose a δ such that the magnitude of the model reduction error is approximately constant over the entire frequency range, then, by equations (5.117)–(5.119), the upper bound of the model reduction error is equal to σ_n which is one half of that obtained by using either Moore's method or Prakash and Rao's method.

In order to illustrate this proposed technique and to verify the new formulae in the theorems and lemmas introduced in this section, an example is presented in Section A of the next chapter.

C. ORDER REDUCTION ALTERNATIVES

As mentioned before, a low order controller is always desirable for ease of implementation and reduced computational load. They are generally two ways to achieve this objective. The first is to reduce the order of the plant's model and then design a controller using the lower order model of the plant. The second way is to design a controller using the high order plant model and then reduce the order of the controller. Both of these ways will be discussed in a sequence.

1. Plant Model Reduction : The idea is to approximate an n^{th} order plant as a r^{th} order plant so that a r^{th} order controller can be designed. The question of how good the closed-loop performance is when the r^{th} order controller is in place with the n^{th} order plant will depend on how good our model reduction technique is. Any of the techniques presented in Section A may be used to reduce the order of the plant.

For any controller design technique, the plant will be an input to the algorithm. Hence, a lower order plant will certainly reduce the computational load. On the contrary, even though the low order controller after controller model reduction is always stable, the stability of the closed-loop system is not guaranteed. For these reasons, plant model reduction is preferrable over controller model reduction.

2. Controller Model Reduction : As explained above, reducing the order of the controller may drive the closed-loop system into instability. Hence, controller model reduction should not be applied without careful deliberations.

Enns (1984b) proposed a method to reduce the order of the controller while maintaining closed-loop stability. He proposed that the reduction error be represented as an additive perturbation as shown in Figure 13.

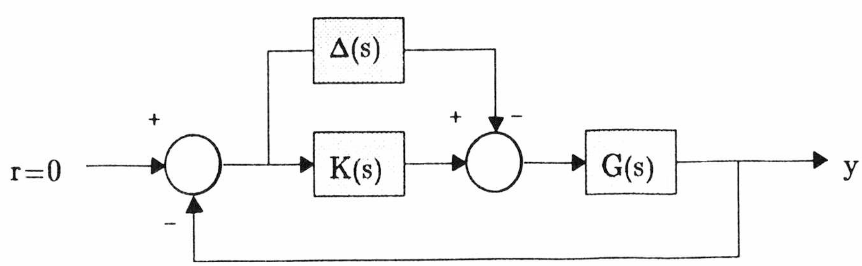

Figure 13 Reduction error as a perturbation

The controller model reduction error is given by

$$\Delta(s) = K(s) - K_r(s) \; . \tag{5.138}$$

It is easily verified that the transfer function seen from $\Delta(s)$ is $(I+GK)^{-1}G$ and hence, by the Small Gain Theorem, the closed-loop in Figure 13 is stable if

$$\| (I+GK)^{-1}G(K-K_r)\|_{\infty} < 1 \; . \tag{5.139}$$

We shall assume that the frequency weighted balanced model reduction technique proposed by Enns (1984a,b) is used for the controller model reduction. Comparing (5.139) with (5.19), we immediately obtain

$$W_o = (I+GK)^{-1}G \quad \text{and} \quad W_i = I \; . \tag{5.140}$$

Thus, with the reduced order controller and full order plant, closed-loop stability is guaranteed if

$$E_{\infty} \triangleq \| W_o (K-K_r) W_i \|_{\infty} < 1 \tag{5.141}$$

where W_o and W_i are given in (5.140).

We summarize this result in the following algorithm for designing a reduced order controller.

<u>Algorithm 5.1</u>

Step 1 : Design a full order controller for the given plant.

Step 2 : Obtain a reduced order model $K_r(s)$ of $K(s)$ using the frequency weighted balanced model reduction method with input and output weighting functions chosen as

$$W_i(s) = I \, , \qquad W_o(s) = (I+GK)^{-1}G \; .$$

Step 3 : Check that $E_{\infty}<1$. If so, then the closed-loop system is guaranteed to be stable with $K_r(s)$ in place of $K(s)$. If $E_{\infty}\geq 1$, then go back to Step 2 and increase the order of $K_r(s)$. Repeat Step 3.

$\triangle$

D. COMBINED STATE- AND OUTPUT FEEDBACK

All the output feedback H_∞ controller design techniques presented in Chapter IV yields a controller with order equal to that of the generalized plant while the state-feedback H_∞ controller is nothing but a constant gain matrix. Also in Chapter IV, we have shown that dynamic state-feedback H_∞ controllers offer absolutely no advantage over static state-feedback H_∞ controllers. These motivated our work on combined state and output feedback H_∞ controller design in order to achieve a lower order H_∞ controller. The central idea is to decompose the plant into two sub-systems where all the states of the first sub-system are available for feedback while the states of the second are not. Two separate controllers are then designed, one for each sub-system, before they are combined. However, by doing so, the dimensions of the input and output of the controller and the plant are no longer compatible which gives rise to the need for the formulation of a composite controller. The closed-loop structure and the performance criteria are modified in this approach.

1. Design Of Controller : Let the generalized plant P(s) be given by

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \tag{5.142}$$

where

$$P_{ij}(s) \in \mathbf{R}(s)^{p_i * m_j}\ , \quad i,j = 1,2 \ \text{ and } \ A \in \mathbf{R}^{n*n}\ . \tag{5.143}$$

It is assumed that the plant P could be transformed into modal form as follows :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} w + \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix} u \tag{5.144a}$$

$$z = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_{11}w + D_{12}u \tag{5.144b}$$

$$y = \begin{bmatrix} C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_{21}w \tag{5.144c}$$

where all the states contained in the vector x_1 are measurable. The system described in (5.144) can be decomposed into two sub-systems, $P_1(s)$ and $P_2(s)$, which are respectively given by

$$\dot{x}_1 = A_{11}x_1 + B_{11}w + B_{21}u \tag{5.145a}$$

$$z_1 = C_{11}x_1 \tag{5.145b}$$

$$y_1 = C_{21}x_1 \tag{5.145c}$$

and

$$\dot{x}_2 = A_{22}x_2 + B_{12}w + B_{22}u \tag{5.146a}$$

$$z_2 = C_{12}x_2 + D_{11}w + D_{12}u \tag{5.146b}$$

$$y_2 = C_{22}x_2 + D_{21}w \tag{5.146c}$$

where $x_1 \in \mathbf{R}^{n_1*1}$, $x_2 \in \mathbf{R}^{n_2*1}$ and $n_1+n_2=n$. Notice that the matrices P_1 and P_2 both have the forms assumed by the state and output feedback H_∞ controller design techniques, respectively. It is immediately clear that the following equations hold :

$$y = y_1 + y_2 \tag{5.147}$$

$$z = z_1 + z_2 \ . \tag{5.148}$$

We then design a static state-feedback H_∞ controller K_1 using y_1 as input and a dynamic output feedback H_∞ controller $K_2(s)$ using y_2 as input. The total controller $K(s)$ is then formed by connecting K_1 and $K_2(s)$ in parallel as depicted in Figure 14. Since we are working only with systems that are linear, we have

$$u = u_1 + u_2 \ . \tag{5.149}$$

Note that K_1 is a zeroth-order gain matrix while the order of K_2 is n_2.

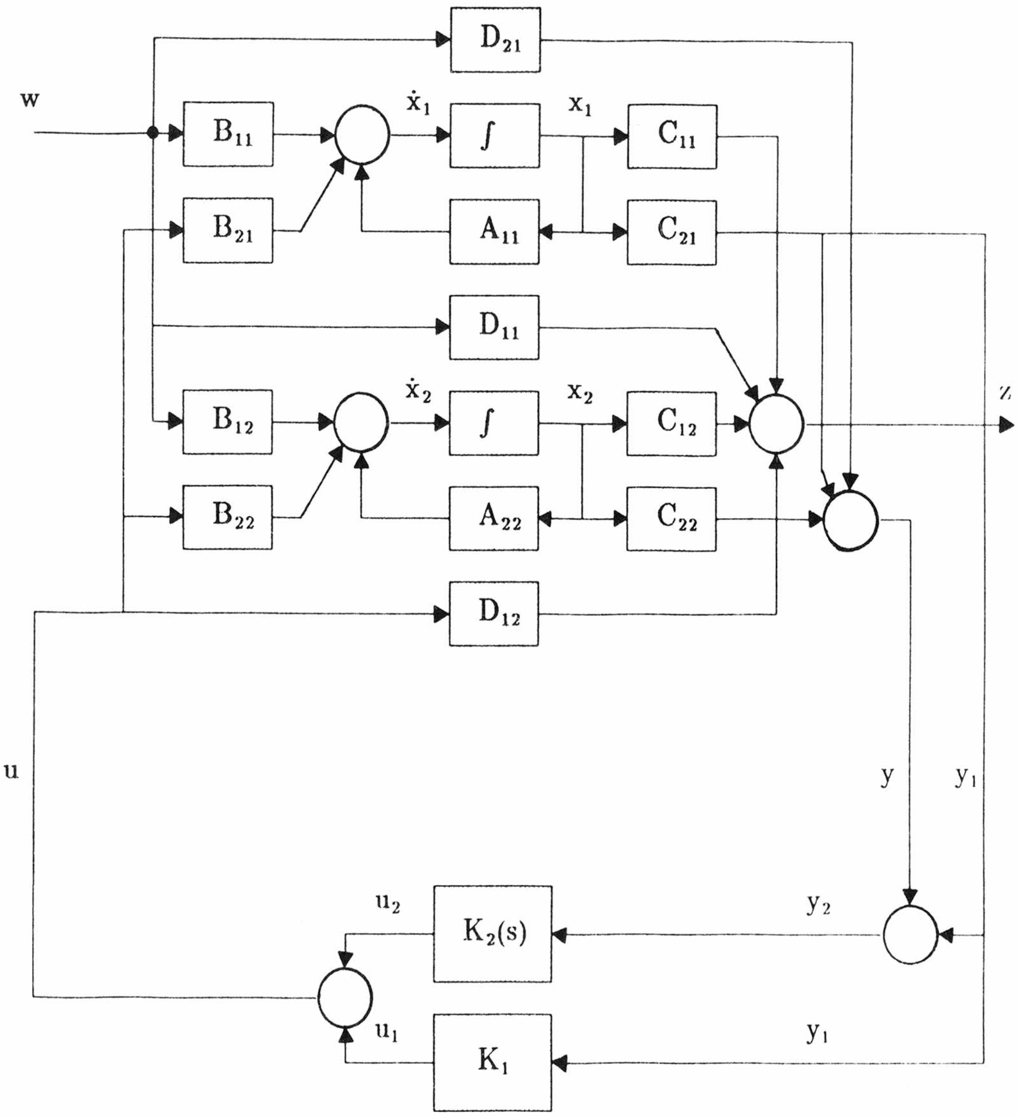

Figure 14 Block diagram for combined state and output feedback

Hence, the order of the combined controller is $n_2 = n - n_1$ and is less than the order of the plant.

2. Composite Controller Formulation : The controller designed in Section 1 requires two input vectors, namely y_1 and y_2 but unfortunately, only the vector y is available as the output of the original plant G(s). Consequently, a special transfer matrix needs to be constructed in order to recover y_1 and y_2. The following theorem serves this purpose.

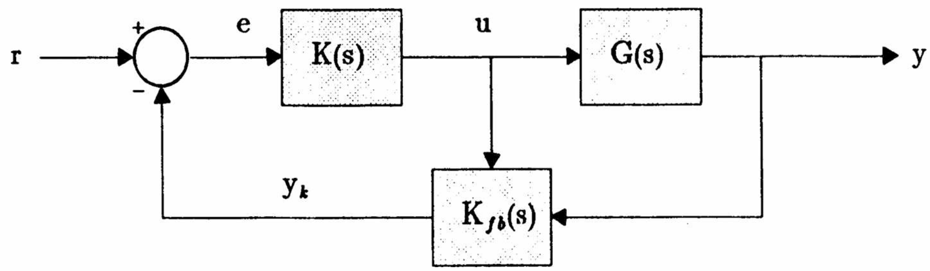

Figure 15 Closed-loop structure with combined state- and output feedback controller

Theorem 5.4

If K(s) in Figure 15 is designed using the technique presented in Section 1 and $y_k = [\, y_1^T \quad y_2^T \,]^T$, then $K_{fb}(s)$ is given by

$$K_{fb}(s) = \left[\, H_2 \quad (\, H_1 - H_2 \,)\, G_1 \right] \tag{5.150}$$

where

$$H_1 = \begin{bmatrix} I_{p2} \\ 0_{p2} \end{bmatrix} \tag{5.151}$$

$$H_2 = \begin{bmatrix} 0_{p2} \\ I_{p2} \end{bmatrix} \tag{5.152}$$

and

$$G_1 = C_{21}\, (\, sI - A_{11}\,)^{-1} B_{21}\ . \tag{5.153}$$

Proof

From equation (5.145), we get

$$y_1 = G_1 u \tag{5.154}$$

and from (5.147), we have

$$y_2 = y - y_1 \,. \tag{5.155}$$

Define

$$\bar{y}_1 \triangleq H_1\, y_1 \tag{5.156}$$

and

$$\bar{y}_2 \triangleq H_2\, y_2 \,, \tag{5.157}$$

then

$$y_k = \bar{y}_1 + \bar{y}_2 \,. \tag{5.158}$$

Combining equations (5.154) — (5.157), we have

$$\begin{aligned} \bar{y}_2 &= H_2\, y - H_2\, y_1 \\ &= H_2\, y - H_2 G_1 u \end{aligned} \tag{5.159}$$

which gives

$$\begin{aligned} y_k &= H_1 G_1 u + H_2\, y - H_2 G_1 u \\ &= \begin{bmatrix} H_2 & (H_1 - H_2)\, G_1 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \\ &= K_{fb}(s) \begin{bmatrix} y \\ u \end{bmatrix} . \end{aligned} \tag{5.160}$$

□

The purpose of the introduction of the matrices H_1 and H_2 is to make the dimension of the feedback signal compatible with the controller input. The block $K_{fb}(s)$ as given in Theorem 5.1 has the structure shown in Figure 16.

Thus, we have presented a way to design a reduced-order H_∞ controller with order n_1 less than the order of the plant at the expense of a more complicated closed-loop structure. Although the structures shown in Figures 5.3 and 5.4 are adequate for control purposes, it would be interesting to transform

Figure 15 to the classical feedback loop as shown in Figure 17.

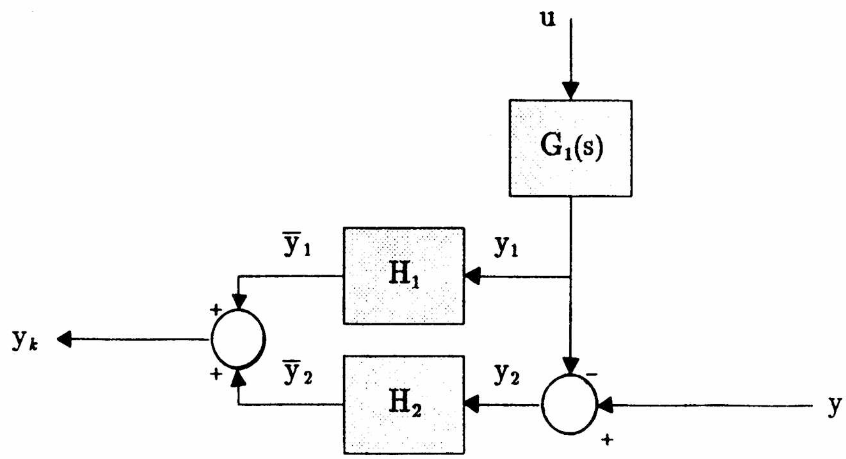

Figure 16 Block structure for $K_{fb}(s)$

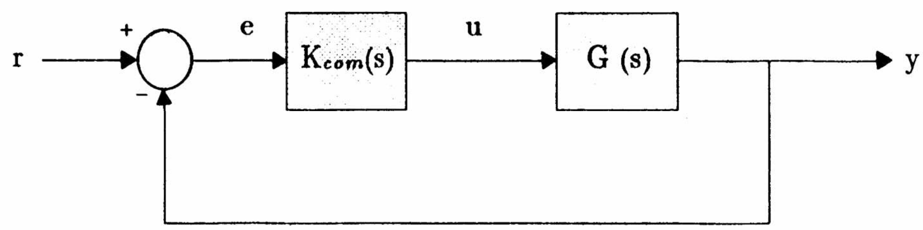

Figure 17 Feedback loop with composite controller

Theorem 5.5

If K(s) in Figure 15 is designed using the technique presented in Section 1, then the composite controller $K_{com}(s)$ in Figure 17 is given by

$$K_{com}(s) = \left[F^{-1}(I+F\,H_2)\,G - G\right]^{-1} \tag{5.161}$$

where

$$F = G\left[I + K\,(H_1 - H_2)\,G_1\right]^{-1} K \tag{5.162}$$

and H_1, H_2 and G_1 are given in (5.151) - (5.153).

Proof

Referring to Figure 15, we have

$$e = r - y_k$$

$$= r - H_2\, y - (\, H_2 - H_2\,)\, G_1 u \tag{5.163}$$

$$u = K\, r - K\, H_2\, y - K\, (\, H_1 - H_2\,)\, G_1 u$$

$$= \left[I + K\, (\, H_1 - H_2\,)\, G_1 \right]^{-1} K\, r - \left[I + K\, (\, H_1 - H_2\,)\, G_1 \right]^{-1} K H_2\, y \tag{5.164}$$

$$y = G\left[I + K\, (\, H_1 - H_2\,)\, G_1 \right]^{-1} K\, r - G\left[I + K\, (\, H_1 - H_2\,)\, G_1 \right]^{-1} K H_2\, y$$

$$= F\, r - F H_2\, y$$

$$= \left[\, (\, I + F H_2\,)^{-1} F\, \right] r \tag{5.165}$$

where the definition of F in (5.162) has been used. Now, from Figure 17, we obtain

$$u = K_{com}\, r - K_{com}\, y \tag{5.166}$$

$$y = G\, K_{com}\, r - G\, K_{com}\, y$$

$$= \left[(\, I + G\, K_{com})^{-1} G\, K_{com} \right] r\, . \tag{5.167}$$

Equating (5.167) with (5.165) gives

$$(\, I + G\, K_{com})^{-1} G\, K_{com} = (\, I + F\, H_2\,)^{-1} F \tag{5.168}$$

$$\left[\, K_{com}^{-1} G^{-1} (\, I + G\, K_{com})\, \right]^{-1} = \left[\, F^{-1} (\, I + F\, H_2\,)\, \right]^{-1} \tag{5.169}$$

$$I + G\, K_{com} = G\, K_{com} F^{-1} (\, I + F\, H_2\,) \tag{5.170}$$

$$(\, G\, K_{com})^{-1} = F^{-1} (\, I + F\, H_2\,) - I \tag{5.171}$$

$$K_{com} = G^{-1} \left[\, F^{-1} (\, I + F\, H_2\,) - I\, \right]^{-1} \tag{5.172}$$

$$K_{com} = \left[F^{-1} (\, I + F\, H_2\,)\, G - G \right]^{-1} . \tag{5.173}$$

□

For the sake of completeness, we also provide the formula for the composite controller that corresponds to Figure 14 which is modified as shown below.

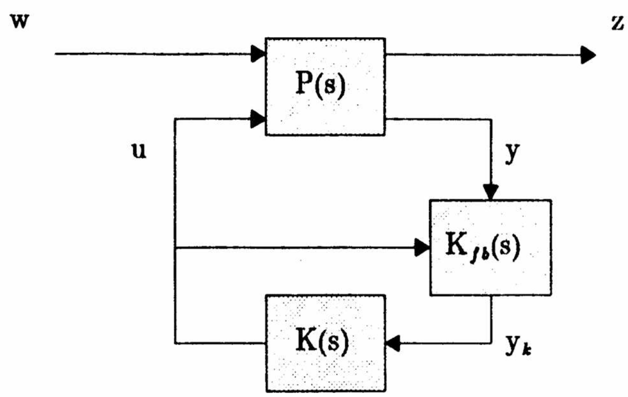

Figure 18 Composite controller for combined state- and output feedback

From equation (5.160), we have

$$y_k = H_2\, y + (\, H_1 - H_2\,)\, G_1 u$$

$$= H_2\, y + (\, H_1 - H_2\,)\, G_1 K\, y_k$$

$$= \left[\, I - (\, H_1 - H_2\,)\, G_1 K \,\right]^{-1} H_2\, y\ . \tag{5.174}$$

Since $u = K\, y_k$, we obtain

$$u = \overline{K}_{com}\, y \tag{5.175}$$

where

$$\overline{K}_{com} = K\left[\, I - (\, H_1 - H_2\,)\, G_1 K \,\right]^{-1} H_2\ . \tag{5.176}$$

<u>**3. Performance Evaluations :**</u> One of the most common criteria used to evaluate the performance of a closed-loop system is its sensitivity to disturbances. The block diagram used is shown in Figure 19.

To evaluate the sensitivity of the output to disturbance d, we set the reference input r to zero. The function of the pre-filter R is to make the dimension of the reference input compatible with the fedback signal. With y_k given by (5.174), we have

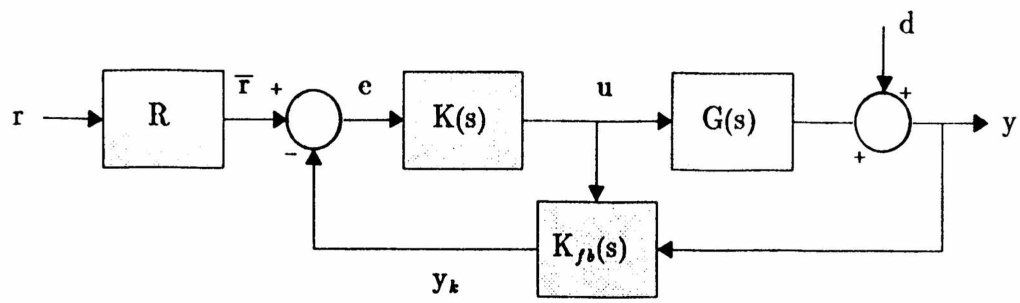

Figure 19 Block diagram for sensitivity definition

$$u = -K\,H_2\,y - K(\,H_1 - H_2\,)\,G_1 u \tag{5.177}$$

$$= -\left[I + K\,(\,H_1 - H_2\,)\,G_1\right]^{-1} K\,H_2\,y \tag{5.178}$$

$$y = d - G\left[I + K\,(\,H_1 - H_2\,)\,G_1\right]^{-1} K\,H_2\,y \tag{5.179}$$

$$= d - F\,H_2\,y \tag{5.180}$$

$$= (\,I + F\,H_2\,)^{-1} d \tag{5.181}$$

where F is given in (5.162). So, we define the sensitivity function as

$$S \triangleq (\,I + F\,H_2\,)^{-1}\,. \tag{5.182}$$

The ability of the system to follow the command input depends on the closed-loop transfer function (with d=0) which is given in (5.165) as

$$y = \left[(\,I + F\,H_2\,)^{-1} F\,R\right] r\,. \tag{5.183}$$

So, the closed-loop transfer function with dimension $p_2 x p_2$ is described by

$$T = S\,F\,R\,. \tag{5.184}$$

We know very well that the closed-loop transfer function is the complement of the sensitivity function, i.e.

$$T = I - S \,, \tag{5.185}$$

hence,

$$\begin{aligned} S\,F\,R &= I - (\,I + F\,H_2\,)^{-1} \\ &= (\,I + F\,H_2\,)^{-1}\big[\,(\,I + F\,H_2\,) - I\,\big] \\ &= S\,F\,H_2 \end{aligned} \tag{5.186}$$

$$\Rightarrow \qquad R = H_2 \tag{5.187}$$

which gives

$$T = S\,F\,H_2 \,. \tag{5.188}$$

It is interesting to point out that this closed-loop transfer function looks very similar to the classical closed-loop transfer fuction which is given by

$$T_c = (\,I + GK\,)^{-1}GK \,. \tag{5.189}$$

The poles of the closed-loop transfer function are the union of the poles of S and the poles of F but unfortunately, neither the stability of the function F nor the minimum phaseness of the function ($I+FH_2$) could be guaranteed. Consequently, this proposed technique ***does not*** ensure closed-loop stability. The investigation of the closed-loop stability of this method should be an interesting research topic.

VI. NUMERICAL EXAMPLES

This chapter contains two examples. In Section A, the proposed balanced truncation model reduction technique with reduced error bound is used to obtain a reduced order model for a large turbo-generator. The theorems and formulae derived in Section B of Chapter V are verified. The reduced order model obtained with δ=0.7 and δ=1 are compared. The examples presented in Section B compare the differences between the two reduced order models mentioned above in the aspect of controller designing.

A. MODEL REDUCTION

This state-space model is taken from Maciejowski (1989). It is a two-input two-output six-state model for a large turbo-generator. The model's matrices are given as follows.

$$A = \begin{bmatrix} -18.4456 & 4.2263 & -2.2830 & 0.2260 & 0.4220 & -0.0951 \\ -4.0977 & -6.0706 & 5.6825 & -0.6966 & -1.2246 & 0.2873 \\ 1.4449 & 1.4336 & -2.6477 & 0.6092 & 0.8979 & -0.2300 \\ -0.0093 & 0.2302 & -0.5002 & -0.1764 & -6.3152 & 0.1350 \\ -0.0464 & -0.3489 & 0.7238 & 6.3117 & -0.6886 & 0.3645 \\ -0.0602 & -0.2361 & 0.2300 & 0.0915 & -0.3214 & -0.2087 \end{bmatrix} \tag{6.1a}$$

$$B = \begin{bmatrix} -0.2748 & 3.1463 \\ -0.0501 & -9.3737 \\ -0.1550 & 7.4296 \\ 0.0716 & -4.9176 \\ -0.0814 & -10.2648 \\ 0.0244 & 13.7943 \end{bmatrix} \tag{6.1b}$$

$$C = \begin{bmatrix} 0.5971 & -0.7697 & 4.8850 & 4.8608 & -9.8177 & -8.8610 \\ 3.1013 & 9.3422 & -5.6000 & -0.7490 & 2.9974 & 10.5719 \end{bmatrix} \tag{6.1c}$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} . \tag{6.1d}$$

Using the proposed model reduction technique as defined in (5.29)–(5.33), the fifth order reduced model is given by

$$A_r = \begin{bmatrix} -0.2085 & 0.3225 & -0.0910 & -0.2353 & 0.2458 \\ -0.3647 & -0.6893 & 6.3113 & 0.7278 & -0.3563 \\ -0.1351 & -6.3153 & -0.1765 & -0.4994 & 0.2287 \\ 0.2353 & 0.9211 & 0.6216 & -2.7734 & 1.6656 \\ -0.3022 & -1.2902 & -0.7317 & 6.0378 & -6.7273 \end{bmatrix} \tag{6.2a}$$

$$B_r = \begin{bmatrix} 0.0250 & 13.7869 \\ 0.0809 & 10.2702 \\ -0.0717 & 4.9188 \\ 0.1701 & -7.6026 \\ 0.0074 & 9.8625 \end{bmatrix} \tag{6.2b}$$

$$C_r = \begin{bmatrix} -8.8629 & 9.8081 & -4.8660 & -4.8332 & 0.6736 \\ 10.5606 & -3.0470 & 0.7225 & 5.8693 & -9.8391 \end{bmatrix} \tag{6.2c}$$

$$D_r = \begin{bmatrix} -0.0062 & 0.0713 \\ -0.0323 & 0.3703 \end{bmatrix} \tag{6.2d}$$

with A_{22} and σ_n take on the values

$$A_{22} = -18.4456 \qquad \text{and} \qquad \sigma_n = 0.2704 . \tag{6.3}$$

Since A_{22} is negative, by Theorem 5.2 and Table I, we should choose $0<\delta<1$. The value of δ chosen here is

$$\delta = 0.7 . \tag{6.4}$$

With this value of δ, the matrix Q as defined in (5.36) is indeed positive definite and thus ensuring the stability of the reduced order model. Also, this reduced order model turns out to be a minimal representation. The eigenvalues of the original and reduced order models are compared in Table II. It is seen that the dominant poles of the reduced order model are very close to those of the original model.

Table II Comparison of eigenvalues

Original Model	Reduced Model
−0.2346+j0.0000	−0.2343+j0.0000
−1.0444+j0.0000	−1.0519+j0.0000
−0.3493+j6.3444	−0.3492+j6.3432
−0.3493−j6.3444	−0.3492−j6.3432
−10.3872+j0.0000	−8.5905+j0.0000
−15.8730+j0.0000	N/A

The spectral norm of the model reduction error is plotted in Figure 20. As predicted by Lemma 5.1, as ω goes to infinity, the spectral norm of the model reduction error is equal to

$$\lim_{\omega \to \infty} \bar{\sigma}[\,\Delta(j\omega)\,] = 2\,\delta\sigma_n = 0.3785\,. \tag{6.5}$$

Furthermore, according to Lemma 5.2, the spectral norm of the model reduction error at zero frequency is equal to

$$\bar{\sigma}[\,\Delta(0)\,] = 0.1491 \tag{6.6}$$

and this is verified by the plot in Figure 20.

The multiple plot in Figure 21 represents the spectral norm of the model reduction error for different values of δ, starting with 0 and ending with 1 with an increment of 0.1. The plot with zero error at zero frequency corresponds to the choice δ=1. Lemma 5.3 asserts that for the special case where c=d which corresponds to δ=1, the model reduction error is bound by

$$\bar{\sigma}[\,\Delta\,]_{c=d} \leq 2\,\sigma_n = 0.5408 \tag{6.7}$$

and accrding to Lemma 5.4,

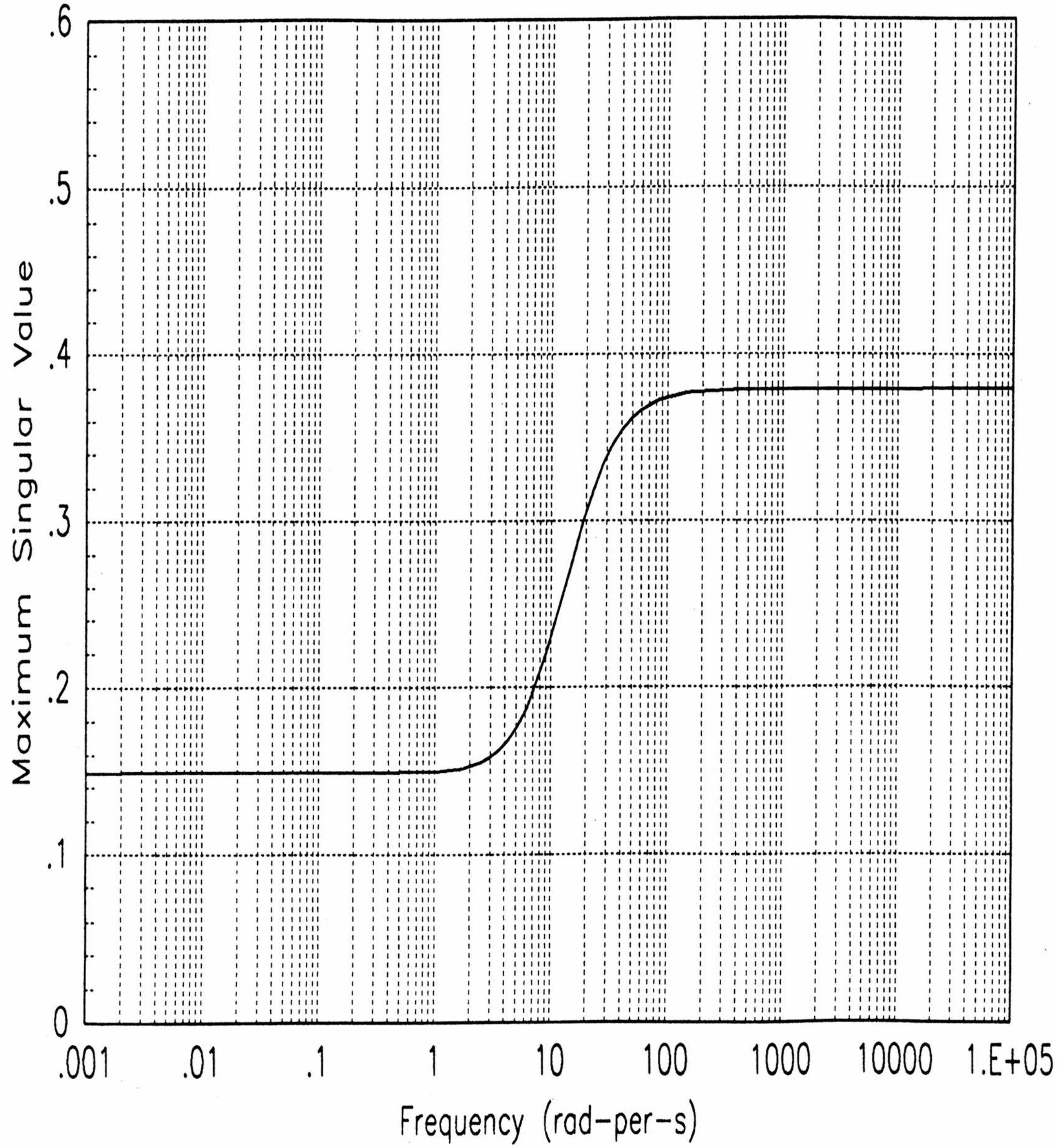

Figure 20 Model reduction error (δ=0.7)

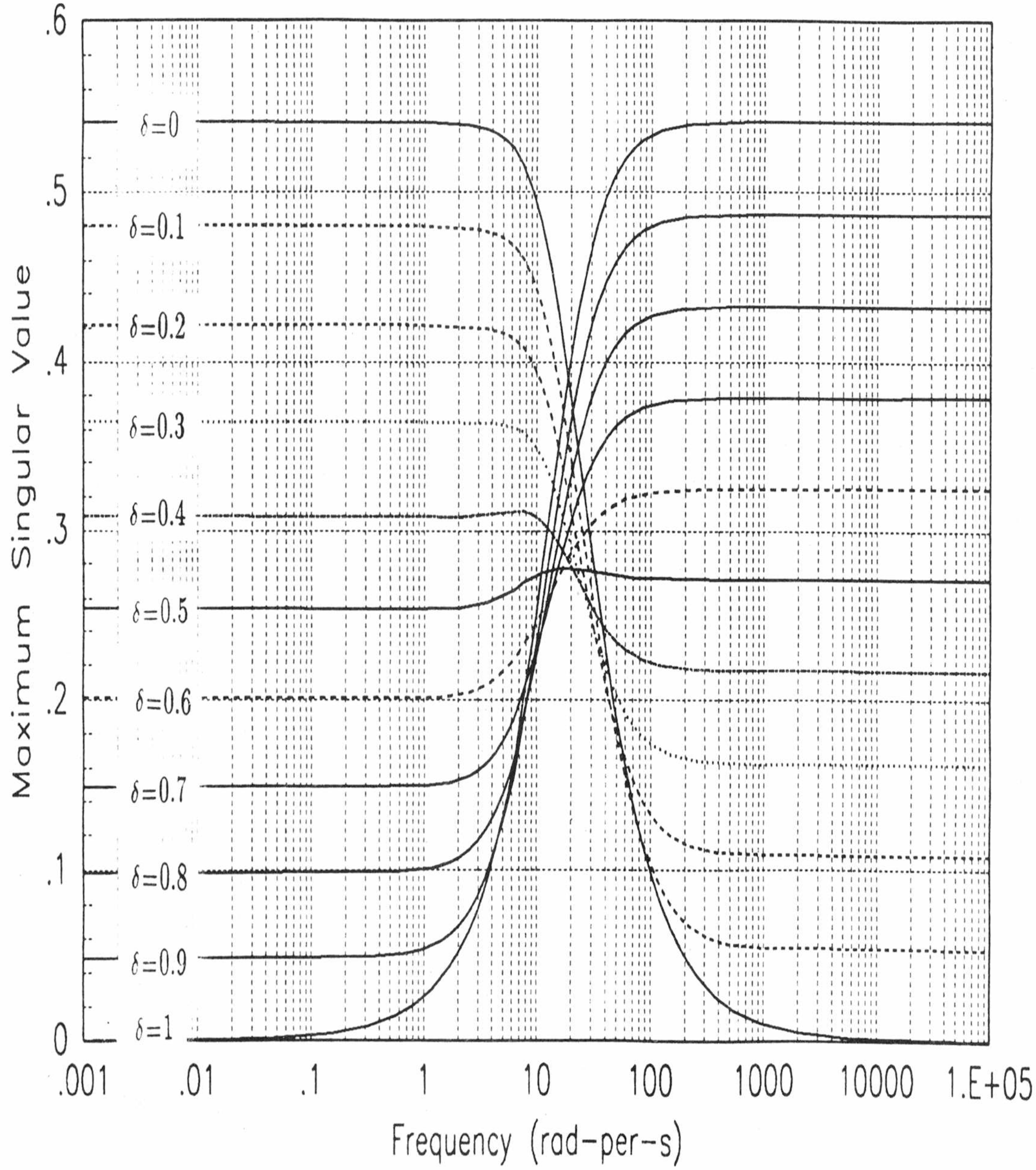

Figure 21 Model reduction error for different values of δ

$$\lim_{\omega \to \infty} \bar{\sigma}[\,\Delta(j\omega)\,] = \bar{\sigma}[\,D - D_r\,] = 0.3785\,. \tag{6.8}$$

Both of these lemmas are verified by the plots in Figure 21. In addition, the claims in (5.94) and (5.117)–(5.119) are supported by this example for all values of δ between 0 and 1.

For the sake of comparison, we also provide some details about the reduced order model obtained using the method proposed by Prakash and Rao (1989) which corresponds to the choice δ=1. For ease of reference, we will refer to this model as $G_r(1)$ and the model obtained with δ=0.7 as $G_r(0.7)$. The poles of $G_r(1)$ are

$$-0.2345 \;,\; -1.0456 \;,\; -0.3499 \pm j6.3436 \;,\; -8.9306 \tag{6.9}$$

which are close to those of $G_r(0.7)$ as given in Table II. As mentioned before, $G_r(0.7)$ is not internally balanced. However, a balanced representation for $G_r(0.7)$ can always be obtained. The gramian for the balanced $G_r(0.7)$ together with the gramian for $G_r(1)$ are tabulated in Table III.

Table III Gramians of $G_r(0.7)$ and $G_r(1)$

$G_r(0.7)$	$G_r(1)$
455.8927	455.8929
76.5111	76.5116
68.5566	68.5571
10.4246	10.4279
7.2301	7.2370

How well the reduced order model approximates the original model is one of the main concerns of any model reduction techniques. As can be deduced from Figure 21, $G_r(0.7)$ is a better approximate of G than $G_r(1)$ at high frequency but the reverse is true for low frequency. In order to compare $G_r(0.7)$ and $G_r(1)$, the

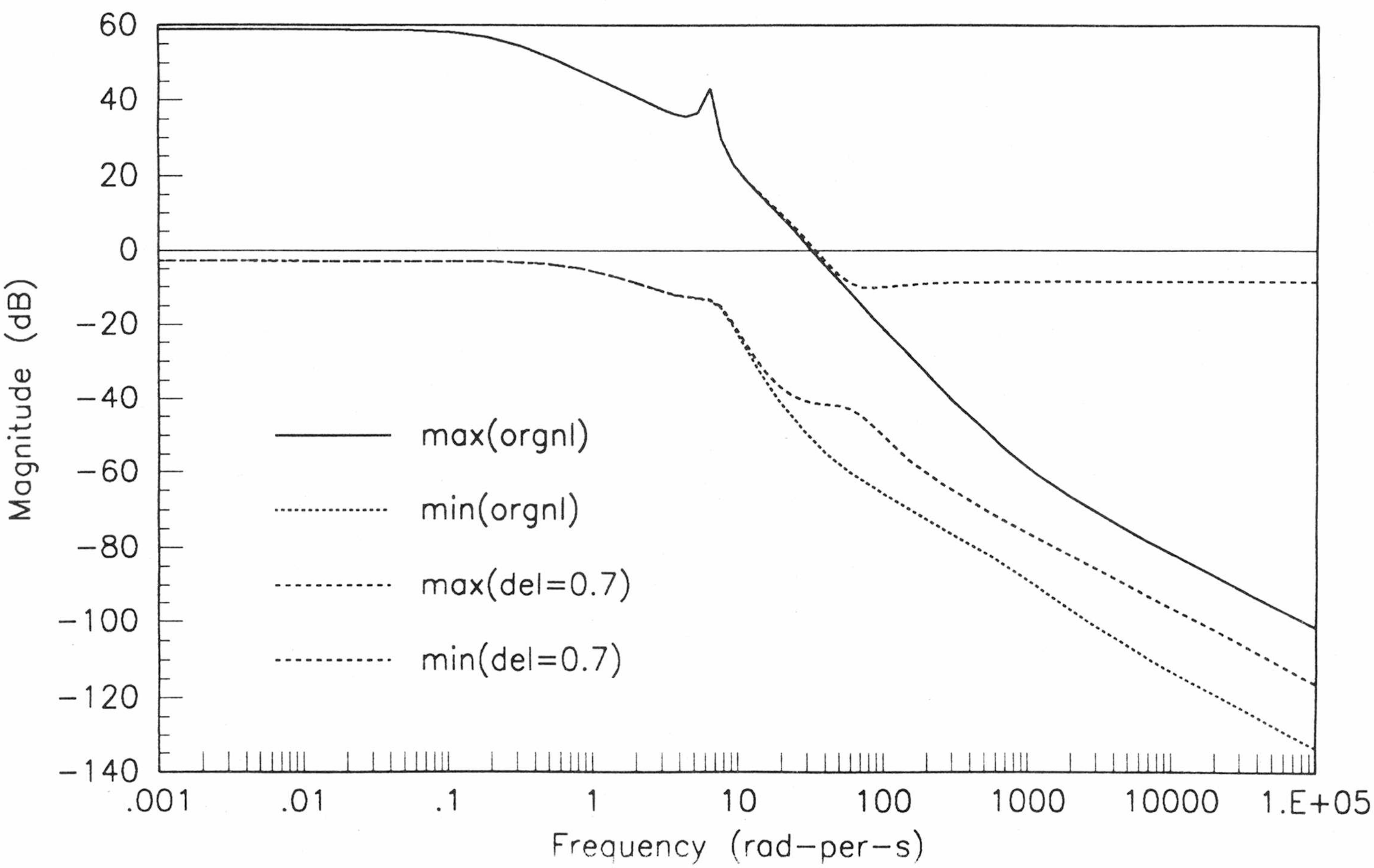

Figure 22 Spectral norms of original and reduced models (δ=0.7)

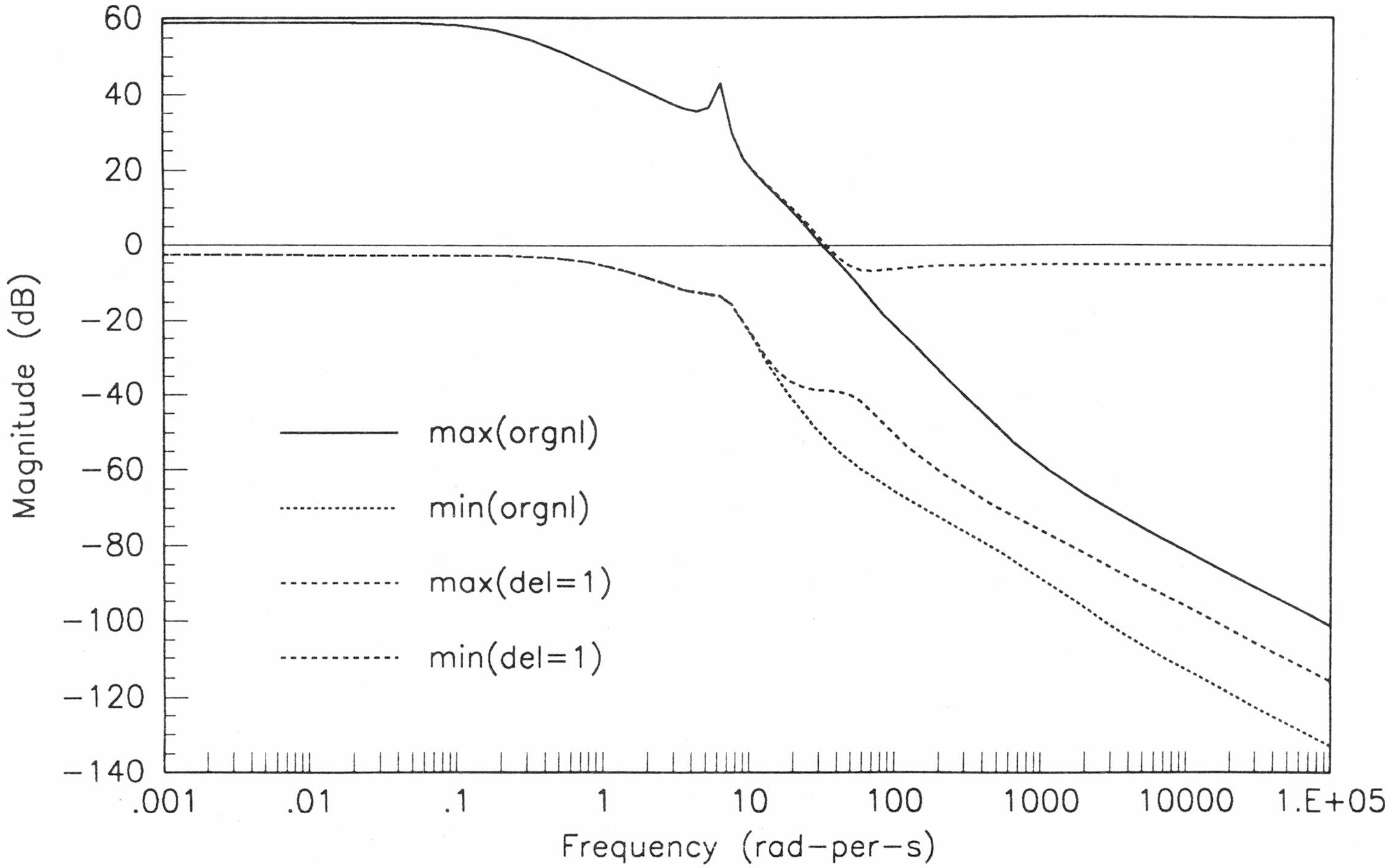

Figure 23 Spectral norms of original and reduced models (δ=1)

spectral norms of $G_r(0.7)$ and G and the spectral norms of $G_r(1)$ and G are plotted in Figures 6.3 and 6.4, respectively. It is seen that, although with a larger model reduction error, $G_r(0.7)$ does approximate G as good as $G_r(1)$ does at low frequencies. On the other hand, at high frequencies, $G_r(0.7)$ does manifest itself as a slightly better approximation of G than $G_r(1)$ is.

B. CONTROLLER DESIGN USING REDUCED ORDER MODEL

The general distance problem approach is used to design an H_∞ controller for the reduced order models obtained in Section A above. This design method is chosen because of its popularity and its ability to solve the H_∞ control problem in its most general form as disscussed in Chapter IV.

We shall pose our problem as the H_∞ problem with combined stability and performance requirements as given in Section B.2 of Chapter III. The weighting matrices are

$$W_1 = w_1 I_2 \qquad \text{and} \qquad W_2 = w_2 I_2 \tag{6.10}$$

where the scalar transfer functions w_1 and w_2 are chosen as

$$w_1 = \frac{(s+5)(s+6)}{(s+0.002)(s+0.4)} \qquad \text{and} \qquad w_2 = \frac{(s+10)(s+50)}{(s+1000)^2} \quad . \tag{6.11}$$

For ease of reference, we denote the controllers obtained for $G_r(0.7)$ and $G_r(1)$ as $K_r(0.7)$ and $K_r(1)$, respectively, where $G_r(0.7)$ and $G_r(1)$ are defined in the previous section. Again, for ease of reference, we denote the four cases of closed-loop system as follows :

case 1 : $K_r(0.7)$ with $G_r(0.7)$

case 2 : $K_r(0.7)$ with G

case 3 : $K_r(1)$ with $G_r(1)$

case 4 : $K_r(1)$ with G .

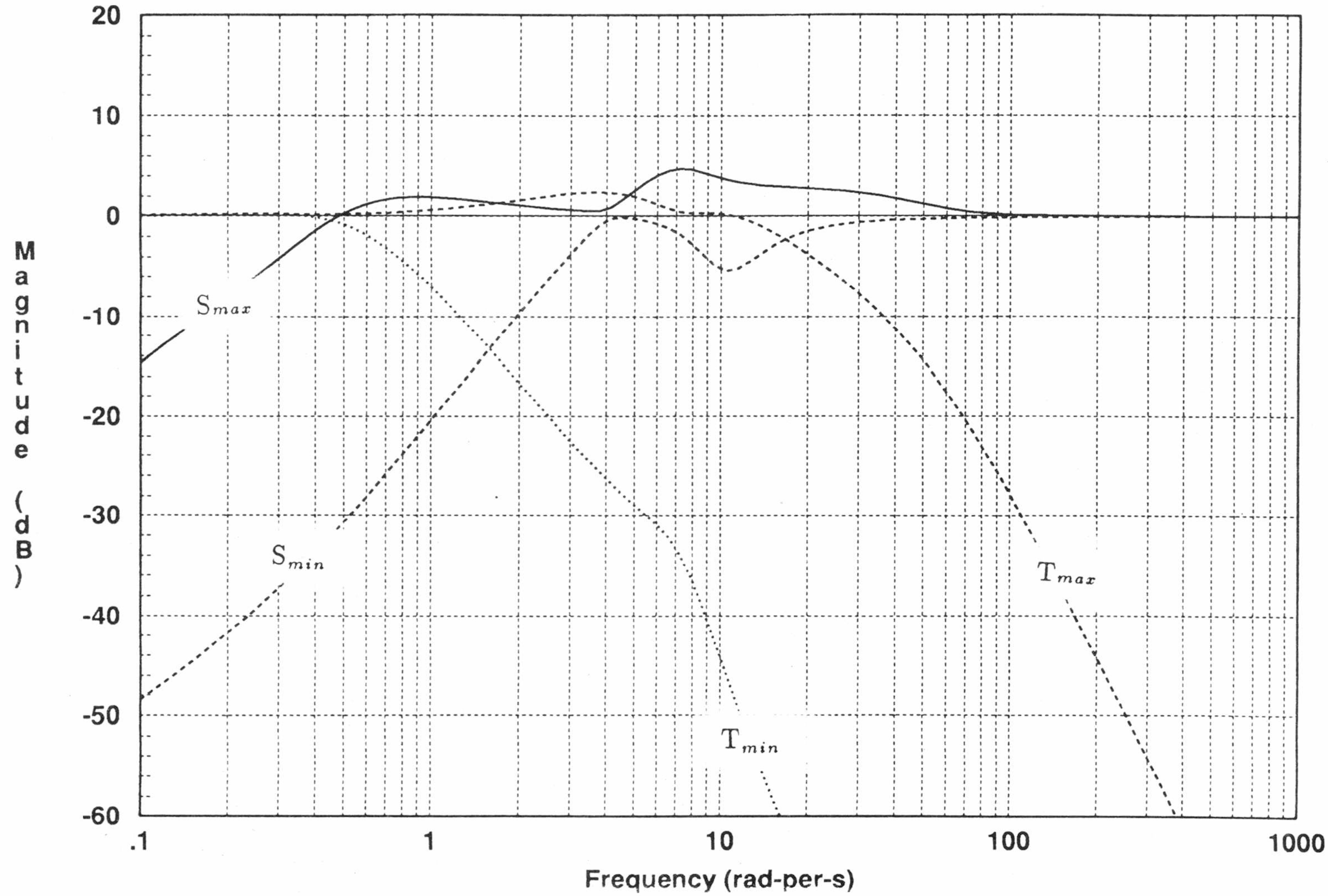

Figure 24 Sensitivity and closed-loop transfer function for case 1

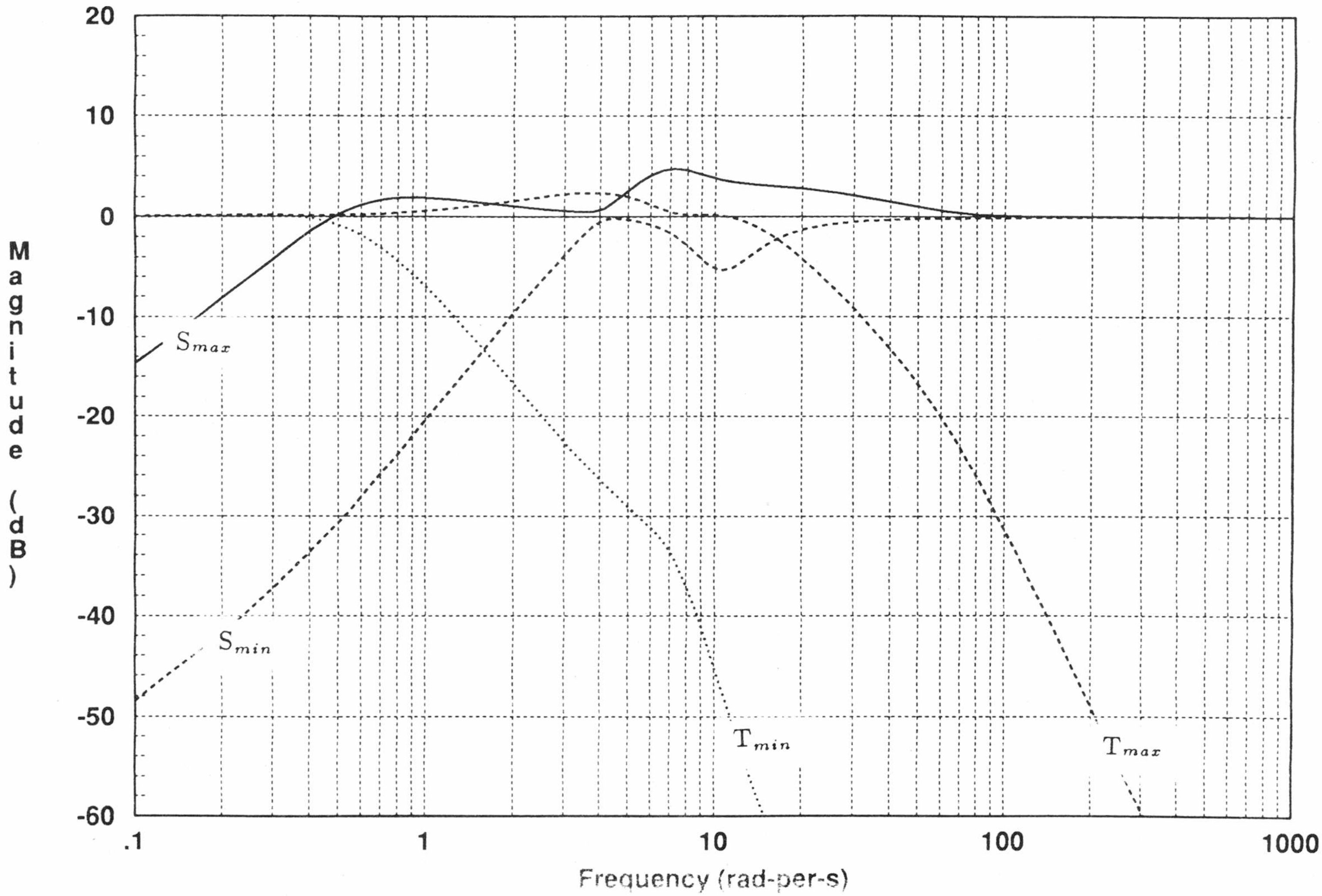

Figure 25 Sensitivity and closed-loop transfer function for case 2

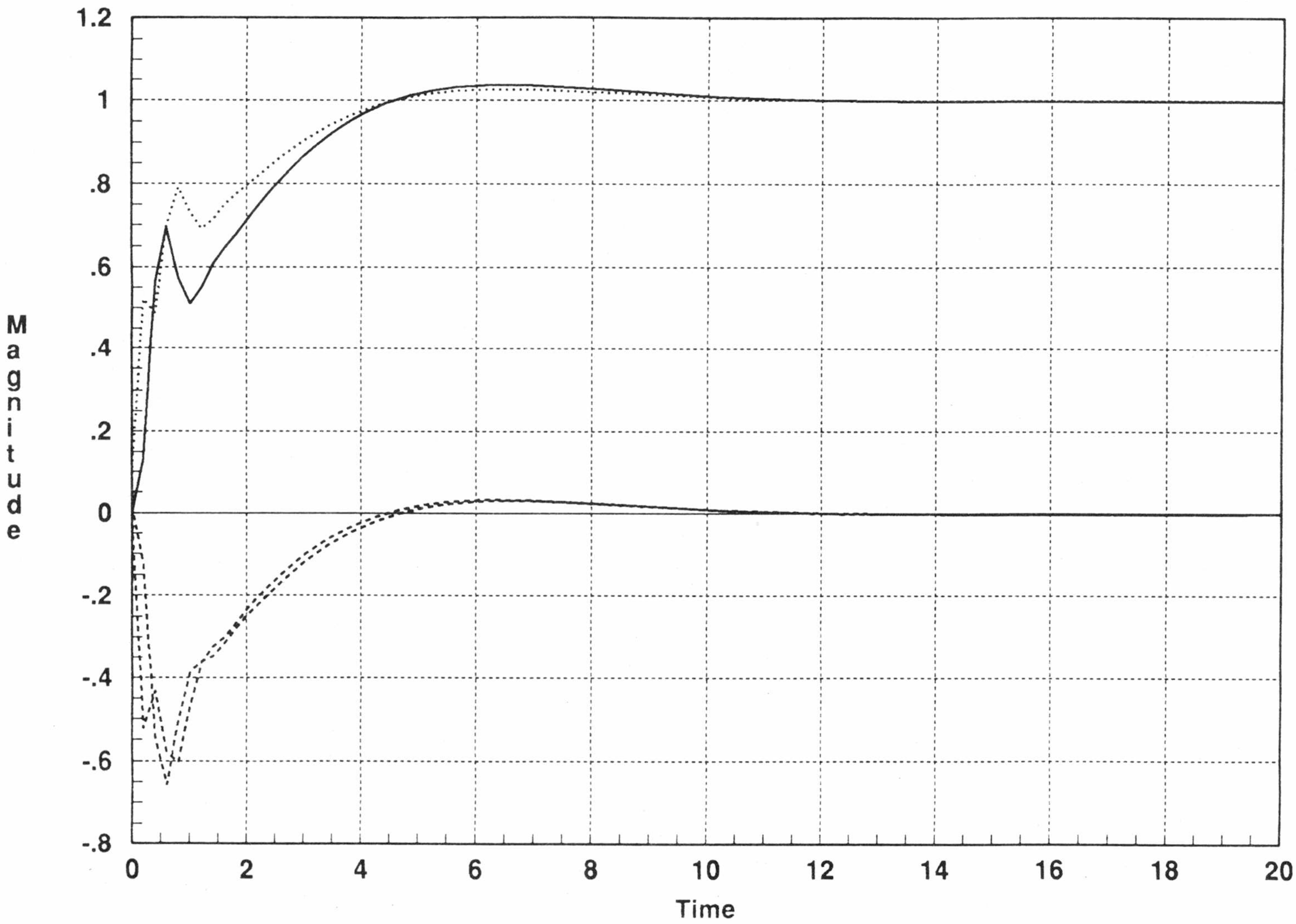

Figure 26 Closed-loop step responses for both cases 1 and 2

The spectral norms of the sensitivity and closed-loop transfer function for case 1 and case 2 are plotted in Figures 24 and 25 respectively. Although the two figures have no perceivable difference for frequencies below 20 rad/s, however, there is a very slight mismatch at high frequencies. This result is expected because the model reduction error is larger at high frequencies. The step responses for both cases 1 and 2 are identical and is shown in Figure 26.

The spectral norms of the sensitivity and the closed-loop transfer function for cases 3 and 4 are shown in Figures 6.8 and 6.9, respectively. It should be pointed out that the mismatch at high frequencies for these two cases is slightly larger than the mismatch for cases 1 and 2. Again, this result is expected because the model reduction error for $G_r(0.7)$ at high frequencies is lower than that of $G_r(1)$. The step responses for cases 3 and 4 are also identical and is shown in Figure 20. These responses are slower that the responses for cases 1 and 2.

The results of this example show that the reduced order model obtained by using the proposed technique does closely approximate the original model. Furthermore, these results raise an interesting point that although $G_r(0.7)$ has a much lower model reduction error than $G_r(1)$ at high frequencies, but as far as H_∞ controller design is concerned, the differences between the two reduced order models may not be very significant.

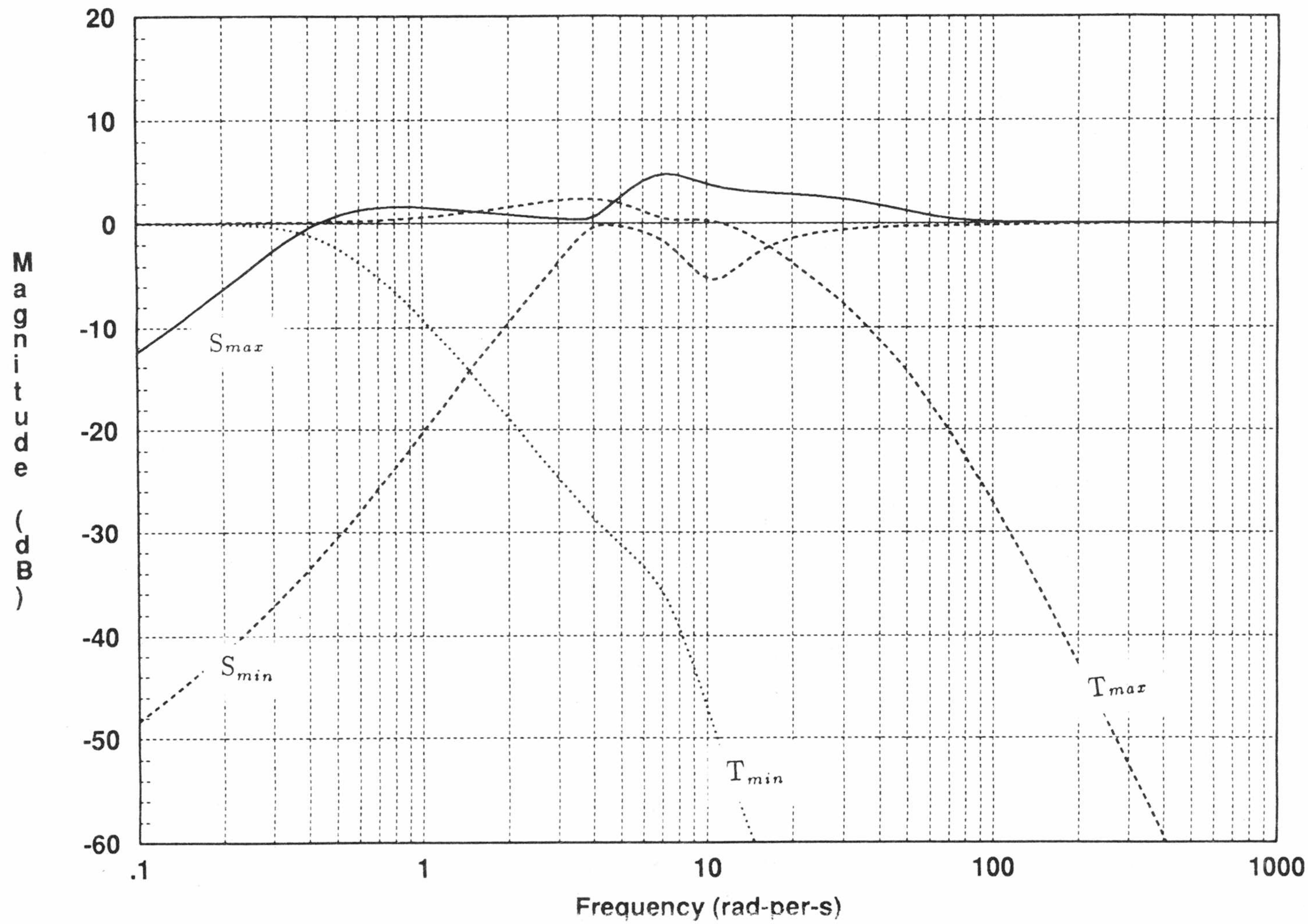

Figure 27 Sensitivity and closed-loop transfer function for case 3

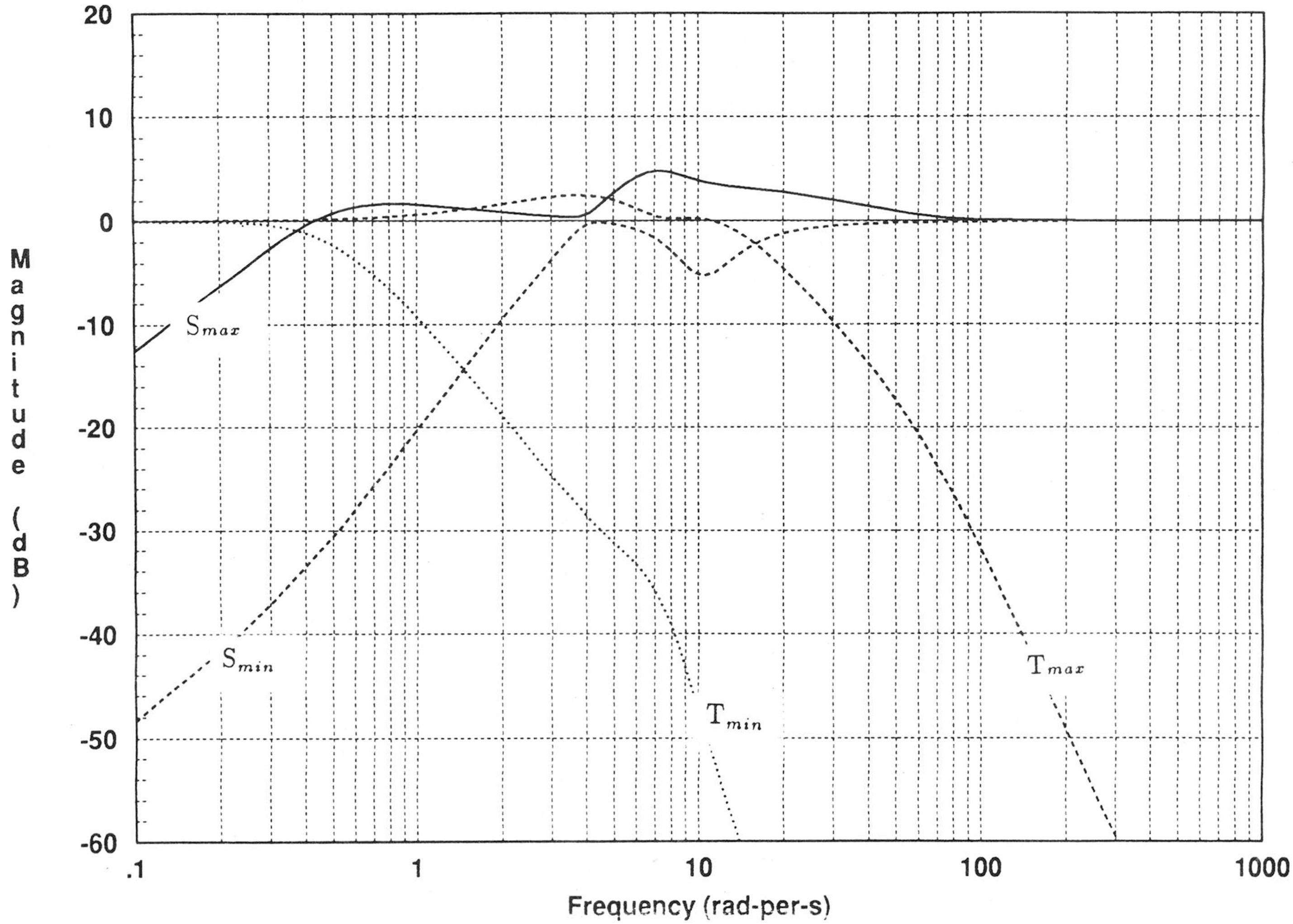

Figure 28 Sensitivity and closed-loop transfer function for case 4

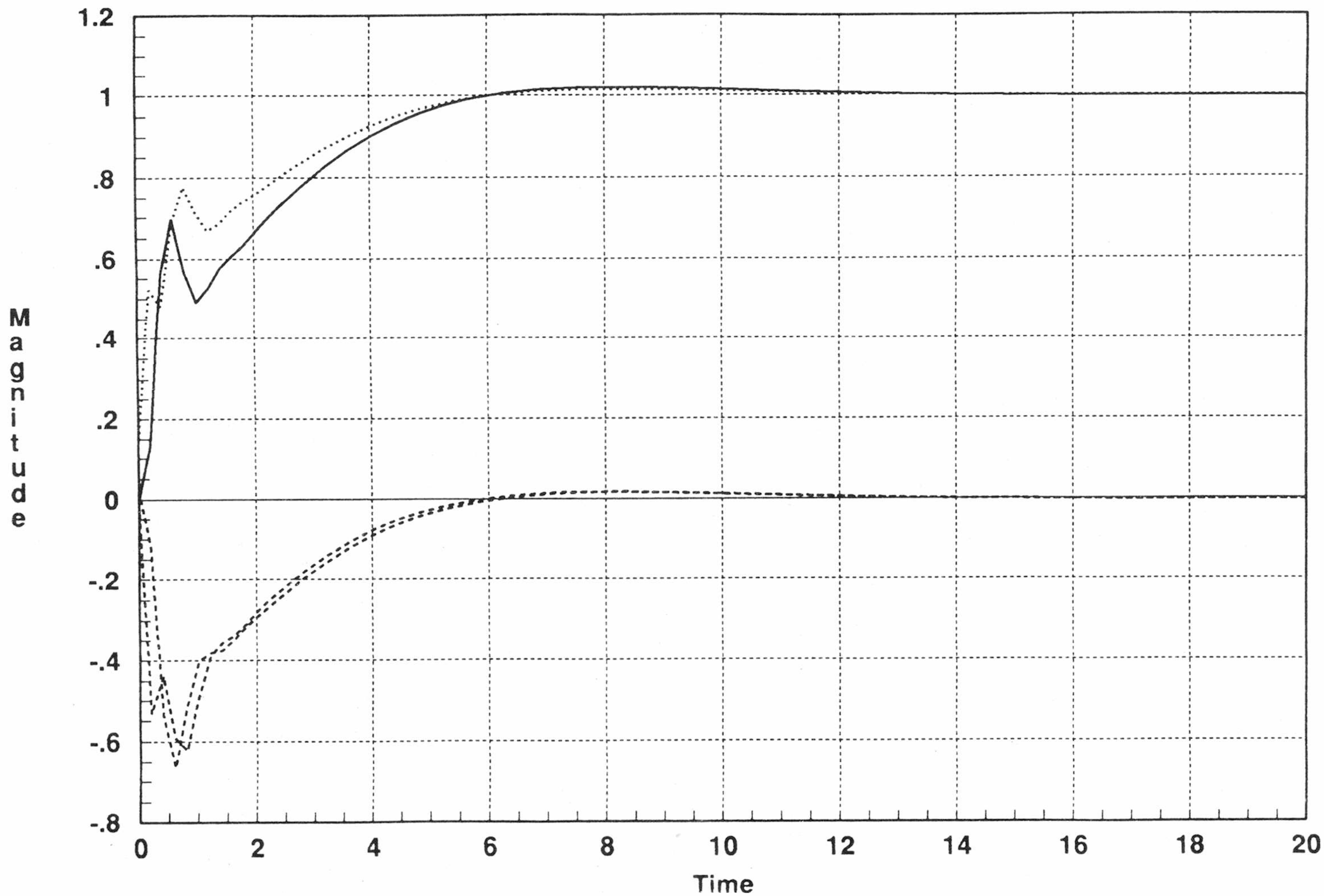

Figure 29 Closed-loop step responses for both cases 3 and 4

VII. CONCLUSION

A. CONCLUDING REMARKS

The balanced truncation with reduced error bound model reduction technique proposed in this thesis offers the designer an extra degree of freedom in which the spectral norm of the model reduction error at low and high frequency ranges can be compromised. Specifically, by properly selecting a value for the new parameter, δ, the relative magnitude of the model reduction error at low and high frequency ranges can be chosen, i.e., by increasing the allowable error at low frequencies by some magnitude, the model reduction error at high frequencies is reduced by the same magnitude. This feature is welcomed because it enables the designer to reduce the upper bound of the model reduction error by as much as 50%.

The fact that the reduced order model obtained by this proposed technique is not internally balanced does not create any calamity because a balanced representation can always be obtained, if needed, once a reduced order model is determined. In the development of this proposed technique, a stability theorem is introduced which provides the condition for which the resulting reduced order model is stable. The absence of the stability guarantee is not a serious drawback because the parameter δ can always assume the values zero or one which will always produce a stable reduced order model.

The combined state and output feedback H_∞ controller design methodology introduced in this thesis provides a direct way to design a reduced order H_∞ controller. This method yields an H_∞ controller with order n_1 less than the order of the generalized plant, where n_1 is the number of measurable states. However, this reduction in controller order is accompanied by a more complicated closed-loop structure. Moreover, the primary assumption made in this method, namely the plant P(s) could be completely decoupled into two sub-systems with all the measurable states contained in the first sub-system, is very restrictive. Many practical systems do not possess this property. Furthermore, no guarantee could be made on the stability of the resulting closed-loop system.

B. SUGGESTIONS FOR FURTHER RESEARCH

The two conditions stated in Theorem 5.3 are necessary but not sufficient to ensure that the model reduction error associated with the proposed model reduction technique is lower than other existing balanced truncation techniques. The derivation of the sufficient condition should be a worthy and stimulating endeavor.

The combined state and output feedback reduced order H_∞ controller design technique proposed in this thesis has the serious drawback that the crucial closed-loop stability requirement is not insured. The investigation of conditions for closed-loop stability should be an interesting research topic.

APPENDIX : SOME MATRIX INVERSION FORMULAE

MATRIX INVERSION LEMMA

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1} \tag{B.1}$$

INVERSE OF BLOCK MATRICES

$$\begin{bmatrix} M & N \\ P & Q \end{bmatrix}^{-1} = \begin{bmatrix} M^{-1}+RS^{-1}T & -RS^{-1} \\ -S^{-1}T & S^{-1} \end{bmatrix} \tag{B.2}$$

where

$$R = M^{-1}N \tag{B.3}$$

$$S = Q - PM^{-1}N \tag{B.4}$$

$$T = PM^{-1}\,. \tag{B.5}$$

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VITA

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During his years as an undergraduate, Mr. Chow had held various temporary positions in several restaurants throughout the country during both summer and winter holidays. He also had worked as Graduate Research Assistant, Graduate Teaching Assistant and Student Teaching Assistant while he was attending graduate school.

Mr. Chow was elected as a member of Tau Beta Pi and Eta Kappa Nu in 1987 and Kappa Mu Epsilon in 1988. He served as the Chairman of Workday Committee for Eta Kappa Nu in Fall 1988. He is also a student member of IEEE since 1987 and is registered in Missouri as Engineer-In-Training in May 1988.